LOCAL COMPLEMENTS TO THE HAUSDORFF-YOUNG THEOREM

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1. INTRODUCTION

Let G be an infinite, locally compact, Abelian group with dual group Γ. For $1 \leq p \leq \infty$, denote by $L^p(G)$ the usual Lebesgue space relative to the Haar measure on G; define $L^p(\Gamma)$ similarly. The Hausdorff-Young theorem [11, Vol. II, p. 227] states that if $1 < p < 2$, then with every function f in $L^p(G)$ there is associated a function $\hat{f}$ in $L^{p'}(\Gamma)$, where p' is the index conjugate to p; the mapping $f \mapsto \hat{f}$ is a bounded linear operator from $L^p(G)$ to $L^{p'}(\Gamma)$, and $\hat{f}$ is the usual Fourier transform of f whenever $f \in L^1(G) \cap L^p(G)$. Accordingly, for $1 \leq p \leq 2$, let

$$FL^p = \{g \in L^{p'}(\Gamma): g = \hat{f} \text{ for some } f \text{ in } L^p(G)\}.$$

For measurable sets $E \subset \Gamma$, denote by $FL^p \mid E$ the set of all functions on E that are restrictions to E of functions in FL^p. Clearly, $FL^p \mid E \subset L^{p'}(E)$. This paper deals with the possibility that $FL^p \mid E \subset L^q(E)$ for some $q \neq p'$.

If E is either finite or locally null [11, Vol. I, p. 124], then all of the spaces $L^q(E)$ for $q < \infty$ coincide. To avoid such trivialities, we assume for the rest of this paper that the set E is infinite and not locally null. In two cases, it follows from the Hausdorff-Young theorem that $FL^p \mid E \subset L^q(E)$ for some $q \neq p'$. First, if Γ is discrete, then

$$FL^p \mid E \subset L^{p'}(E) \subset L^q(E) \qquad \text{for all } q \geq p'.$$

Second, if the Haar measure $|E|$ of E is finite, then

$$FL^p \mid E \subset L^{p'}(E) \subset L^q(E) \qquad \text{for all } q \leq p'.$$

Thus the interest lies in the remaining cases:

(i) Γ is not discrete, and $q > p'$;

(ii) $|E| = \infty$ and $q < p'$.

The following three theorems constitute the main results of this paper.

THEOREM 1. *If* Γ *is not discrete and* E *is not locally null, then*

$$FL^p \mid E \not\subset \bigcup_{q > p'} L^q(E).$$

THEOREM 2. (a) *If* Γ *is not discrete, then* $FL^p \not\subset \bigcup_{q > p'} L^q(\Gamma)$.

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(b) *If* Γ *is not compact, then* $FL^p \not\subset \bigcup_{q<p'} L^q(\Gamma)$.

(c) *If* Γ *is neither compact nor discrete, then* $FL^p \not\subset \bigcup_{q \neq p'} L^q(\Gamma)$.

THEOREM 3. *If* Γ *is not compact, then it contains open sets* E, *of infinite measure, such that for* $1 < p < 2$

$$FL^p \mid E \subset \bigcap_{2 \leq q \leq p'} L^q(E).$$

By Theorem 1, the inclusion relation $FL^p \mid E \subset L^q(E)$ does not hold in case (i); by Theorems 2 and 3, it depends on the set E whether the relation holds in case (ii). We prove Theorems 1 and 2 in Section 2. In Section 3, we use Theorem 1 to give a new proof that if G is not compact and $q < p$, then the only multiplier from $L^p(G)$ to $L^q(G)$ is the 0-operator. Finally, in Section 4, we prove Theorem 3 and obtain as a corollary the known fact that if Γ is infinite and $p \neq 2$, then $FL^p \neq L^{p'}(\Gamma)$.

Theorem 1 is new. Theorem 3 is known for discrete groups Γ (see [16, p. 130]), but is new for the case where Γ is not discrete. Theorem 2 is described as known in [9, p. 81], but there is no complete proof in the literature; we include the theorem here because it follows easily from Theorem 1 and complements Theorem 3. Special cases of Theorem 2 are proved in [1, pp. 261-263] and [4, Vol. II, p. 147].

This paper overlaps slightly with a recent paper of R. E. Edwards [5], although the work on the two papers was done independently. Edwards concentrates on the case where Γ is discrete and infinite, and he shows in this case that if $1 < p < 2$, then FL^p is not contained in the algebraic sum

$$\bigcup_{q<p'} L^q(\Gamma) + \bigcup_{r>p} FL^r.$$

This result and Theorem 1 are complementary, and both are sharper than Theorem 2. I wish to thank Professor Edwards for providing me with a preprint of his paper. I also wish to thank the referee for suggesting a method of extending Theorem 1 to the case where Γ is an infinite, compact, non-Abelian group.

2. NONINCLUSION THEOREMS

First we establish some notation. If g is a function on Γ and E is a subset of Γ, we denote by $g \mid E$ the restriction of g to E, and by $\| g|E \|_p$ the norm of $g \mid E$ as a member of $L^p(E)$; for a continuous function g we denote the support of g by supp g. Sometimes we specify a particular normalization of the Haar measure on Γ; we assume in such cases that the Haar measure on G is also renormalized so that the inversion theorem and the Plancherel theorem [16, pp. 22, 26] hold. Renormalization changes norms of functions but does not affect the containment or lack of containment of $FL^p \mid E$ in $L^q(E)$.

Even if G is not Abelian, we denote the group operation in G by + and the identity by 0. If E and K are subsets of G, then E + K is the set of all sums $x + x'$ with x in E and x' in K; the set E - K is defined similarly.

To prove Theorems 1 and 2, we need the existence, on nondiscrete groups, of positive-definite functions that have small support but are relatively large on relatively large parts of their support.

LEMMA 0. *Let* G *be an infinite, compact group, or a nondiscrete, locally compact, Abelian group. Then there exist a compact set* K *in* G *and a constant* $\delta > 0$ *such that to each* $\varepsilon > 0$ *there corresponds a continuous, positive-definite function* g *on* G *with the properties*

(a) $0 \le g \le 1$ *and* $g(0) = 1$,

(b) $\operatorname{supp} g \subset K$ *and* $|\operatorname{supp} g| < \varepsilon$,

(c) $|\{x \in G\colon g(x) \ge \delta\}| \ge \delta\,|\operatorname{supp} g|$.

Proof. The model for g is the triangular function Δ on the real line R given by

$$\Delta(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, Δ is at least 1/2 on half of its support. Note also that Δ is the convolution square of the characteristic function of the closed interval $[-1/2, 1/2]$.

Functions g with the desired properties exist on every locally compact group G satisfying the following condition.

Condition A. There exist a compact set K in G and a positive constant C such that to each $\varepsilon > 0$ there corresponds a pair $\{U, V\}$ of symmetric, compact neighbourhoods of 0 with the properties

1. $V + V \subset U$,
2. $U + U \subset K$,
3. $|U + U| \le C\,|V| \le \varepsilon$.

Indeed, suppose that the group G satisfies Condition A. Given $\varepsilon > 0$, choose sets U and V as above. Let h be the characteristic function of U, and let

$$g = h * h / |U|.$$

Clearly, the function g is continuous and positive-definite. Since

$$g(x) = |U \cap (x + U)| / |U|,$$

assertion (a) holds. Moreover, g vanishes off the compact set $U + U$; therefore assertion (b) holds. Finally, for each x in V, we have the relation $x + V \subset U$; for such x,

$$g(x) \ge |x + V| / |U|.$$

Now $U \subset U + U$, because $0 \in U$. It follows that if $x \in V$, then

$$g(x) \ge |V| / |U + U| \ge 1/C.$$

Therefore

$$|\{x\colon g(x) \ge 1/C\}| \ge |V| \ge (1/C)|U + U| \ge (1/C)\,|\operatorname{supp} g|.$$

This is assertion (c) with $\delta = 1/C$.

It remains to show that if a nondiscrete group is locally compact and Abelian, or merely compact, then it satisfies Condition A. We suppose first that G is infinite and compact, but possibly not Abelian. Consider the set of continuous, irreducible, unitary representations of G. For each such representation α, of degree d_α, let $\alpha(G)$ be the image of G in the unitary group $U(d_\alpha)$. There are two possibilities:

Case 1. For some α, the group $\alpha(G)$ is infinite.

Case 2. For all α, the group $\alpha(G)$ is finite.

In Case 1, $\alpha(G)$ is an infinite, compact subgroup of $U(d_\alpha)$. Normalise the Haar measure on $\alpha(G)$ so that $\alpha(G)$ and G have the same mass. The proof of [11, Vol. II, p. 651, 44.29] shows that there exist two constants κ and κ' and a sequence $\{U_n, V_n\}_{n=1}^{\infty}$ of pairs of symmetric, compact neighbourhoods of the identity in $\alpha(G)$ such that

(i) $U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$,

(ii) $0 < |U_n + U_n| < \kappa\, |U_n|$,

(iii) every neighbourhood of the identity contains some U_n,

(iv) $V_n + V_n \subset U_n$, but $|U_n| < \kappa' |V_n|$.

In the terminology of [11, Vol. II, p. 637, 44.10], the group $\alpha(G)$ admits a compact, symmetric D''-sequence of neighbourhoods of the identity.

Because $\alpha(G)$ is infinite and compact, it has open sets of arbitrarily small positive measure. Given $\varepsilon > 0$, use properties (iii) and (iv) to obtain an integer n for which $\kappa\kappa' |V_n| \leq \varepsilon$. Then

$$|U_n + U_n| < \kappa\kappa' |V_n| \leq \varepsilon .$$

Returning to the group G, let $K = G$, let $C = \kappa\kappa'$, and let U and V be the inverse images $\alpha^{-1}(U_n)$ and $\alpha^{-1}(V_n)$. Assertions 1 and 2 of Condition A follow immediately from the definitions of the sets involved. As in [5, p. 195], the set mapping α^{-1} is measure-preserving. Therefore, Assertion 3 holds. This completes the proof in Case 1.

In Case 2, for each continuous, irreducible representation α, the kernel G_α is a closed subgroup having finite index in G; hence G_α is open. For each nonzero x in G, there exists such a representation α with $x \notin G_\alpha$ (see [2, p. 75, 2.4], or [11, Vol. II, p. 343, 22.12]). By taking finite intersections of suitable subgroups G_α, we obtain open subgroups having arbitrarily large finite index in G. Given $\varepsilon > 0$, choose such a subgroup H of index at least $|G|/\varepsilon$. Let $U = V = H$. Then Assertions 1 to 3 hold with $K = G$ and $C = 1$. This completes the proof in Case 2.

Now suppose that G is a nondiscrete, locally compact, Abelian group. By the structure theorem ([11, Vol. I, p. 389], or [16, p. 40]), the group G has an open subgroup G_1 of the form $G_1 = L \times R^n$, where L is a compact, Abelian group. Again there are two possibilities:

Case 3. $n = 0$.

Case 4. $n > 0$.

In Case 3, the subgroup G_1 is itself compact; on the other hand, G_1 is infinite because it is open in the nondiscrete group G. Therefore, by Cases 1 and 2, the group G_1, with its own Haar measure, satisfies Condition A. But the Haar measure on G_1 is simply a multiple of the restriction to G_1 of the Haar measure on G; hence G also satisfies Condition A.

In Case 4, let $K = L \times [-1, 1]^n$, and let $C = 4^n$. For $0 < a \leq 1/4$, let $U = L \times [-2a, 2a]^n$ and $V = L \times [-a, a]^n$. Clearly, Assertions 1 and 2 hold, and Assertion 3 holds for all sufficiently small a. This completes the proof of the lemma.

Remark 1. In the original version of this paper, Lemma 0 was stated and proved only for Abelian groups. I wish to thank the referee for observing that the case of compact, non-Abelian groups can be handled in the manner described above. In the rest of this paper, the main theorems will be stated and proved for Abelian groups, and the analogous statements for non-Abelian groups will be discussed in the remarks.

For the rest of Section 2, we consider Abelian groups only, and we use the notation of Section 1.

THEOREM 1. *Suppose that* Γ *is not discrete and that* $E \subset \Gamma$ *is not locally null. Let* $1 < p \leq 2$. *Then* $FL^p \mid E \not\subset \bigcup_{q > p'} L^q(E)$.

Proof. Since E is not locally null, it contains a subset of positive but finite measure; by the inner regularity of Haar measure, this subset contains a compact set of positive measure. It is therefore sufficient to prove the theorem for compact sets E, and we assume henceforth that E is compact and has positive measure.

Suppose that $FL^p \mid E \subset L^q(E)$ for some q in the interval (p', ∞). Then the mapping $f \mapsto \hat{f} \mid E$ taking $L^p(G)$ into $L^q(E)$ is closed. Therefore it is bounded, and there exists a constant C such that, for all f in $L^p(G)$,

$$\| \hat{f} \mid E \|_q \leq C \|f\|_p . \tag{1}$$

Let $\varepsilon > 0$. We apply Lemma 0 to the group Γ, obtaining a continuous, positive-definite function g on Γ, with properties (a), (b), and (c). Let $\lambda = |\mathrm{supp}\, g|$. Because g is continuous, positive-definite, and compactly supported, there exists a positive function f in $L^1(G)$ such that $g = \hat{f}$ [11, Vol. II, p. 297]. Now $\|f\|_1 = \hat{f}(0) = 1$, and $\|f\|_2 = \|g\|_2 \leq \lambda^{1/2}$. Therefore, by the logarithmic convexity of $\|f\|_p$ as a function of 1/p, we have the inequality

$$\|f\|_p \leq \lambda^{1/p'} . \tag{2}$$

Next we consider the restrictions to E of translates of g. For γ in Γ, let $\tau_\gamma g$ be the function given by $\tau_\gamma g(\gamma') = g(\gamma' - \gamma)$ for all γ' in Γ. Consider the integral

$$\int_E [\tau_\gamma g(\gamma')]^q \, d\gamma' = \int_E g(\gamma' - \gamma)^q \, d\gamma' .$$

This is 0, unless for some γ' in E, $\gamma' - \gamma$ lies in the set K provided by Lemma 0. Such γ' exist only if γ lies in the compact set E - K. Thus

$$\int_{E-K} \int_E g(\gamma' - \gamma)^q \, d\gamma' \, d\gamma = \int_\Gamma \int_E g(\gamma' - \gamma)^q \, d\gamma' \, d\gamma$$

$$= \int_E \int_\Gamma g(\gamma' - \gamma)^q \, d\gamma \, d\gamma' = |E| \cdot \|g\|_q^q .$$

It follows that, for some γ in $E - K$,

$$\int_E \tau_\gamma g(\gamma')^q \, d\gamma' \geq (|E|/|E - K|) \, \|g\|_q^q .$$

But, by assertion (c) of Lemma 0, $\|g\|_q^q \geq \delta^{q+1}\lambda$; hence, for some γ in $E - K$,

$$\| \tau_\gamma g \,|\, E \|_q^q \geq \delta^{q+1}\lambda \, |E|/|E - K| . \tag{3}$$

Now, $\tau_\gamma g$ is the transform of the function γf. By inequality (1), applied to γf,

$$\| \tau_\gamma g \,|\, E \|_q \leq C \, \|\gamma f\|_p \leq C\lambda^{1/p'} . \tag{4}$$

Combining inequalities (3) and (4), we see that

$$\delta^{q+1} \, |E|/|E - K| \leq C^q \lambda^{q/p'-1} . \tag{5}$$

Since $\lambda \leq \varepsilon$ and $q > p'$, the right side of (5) tends to 0 as ε tends to 0, but the left side is fixed and positive. This contradiction proves that

$$FL^p \,|\, E \not\subset L^q(E) \quad \text{if } p' < q < \infty . \tag{6}$$

The rest of the proof is an argument by Baire category. For $q \in (p', \infty)$, consider the extended real-valued function ϕ on $L^p(G)$ defined by the equation $\phi(f) = \| \hat{f} \,|\, E \|_q$. By (6), ϕ takes the value $+\infty$; also, ϕ is sublinear and lower-semicontinuous. Therefore $\{f \in L^p(G): \phi(f) = \infty\}$ is a dense set of type G_δ in $L^p(G)$ [17, p. 99]. Choose a strictly decreasing sequence $\{q_n\}_{n=1}^\infty$ converging to p'. By Baire's theorem, the set

$$\{f \in L^p(G): \| \hat{f} \,|\, E \|_{q_n} = \infty \text{ for all } n\}$$

is a dense set of type G_δ in $L^p(G)$. Fix a function f in this set; let $q \in (p', \infty]$. Now $\hat{f} \,|\, E \in L^{p'}(E)$, by the Hausdorff-Young theorem; if also $\hat{f} \,|\, E \in L^q(E)$, then $\hat{f} \,|\, E \in L^{q_n}(E)$ for all sufficiently large n, contrary to the choice of f. Therefore $\hat{f} \,|\, E \not\subset L^q(E)$ for all $q > p'$, and the proof of the theorem is complete.

THEOREM 2. *Let* G *be a locally compact Abelian group with dual group* Γ.

(a) *If* Γ *is not discrete and* $1 < p \leq 2$, *then* $FL^p \not\subset \bigcup_{q > p'} L^q(\Gamma)$.

(b) *If* Γ *is not compact and* $1 \leq p \leq 2$, *then* $FL^p \not\subset \bigcup_{q < p'} L^q(\Gamma)$.

(c) *If* Γ *is neither compact nor discrete and* $1 < p \leq 2$, *then*

$$FL^p \not\subset \bigcup_{q \neq p'} L^q(\Gamma) .$$

Proof. Assertion (a) is simply Theorem 1 with $E = \Gamma$.

To prove (b), suppose first that $p = 2$. Then, by the Plancherel theorem, $FL^2 = L^2(\Gamma)$. If Γ is not compact, then $L^2(\Gamma) \not\subset \bigcup_{q < 2} L^q(\Gamma)$; therefore $FL^2 \not\subset \bigcup_{q < 2} L^q(\Gamma)$.

Next suppose that Γ is not compact and $p < 2$. Suppose that $FL^p \subset L^q(\Gamma)$, for some q in the interval $[2, p')$. By the closed-graph theorem, the mapping $T: f \mapsto \hat{f}$ is bounded from $L^p(G)$ to $L^q(\Gamma)$. Therefore the dual operator T' is bounded from $L^{q'}(\Gamma)$ to $L^{p'}(G)$. It is easy to verify that $T'g = \hat{g}$ for all functions g in $L^1(\Gamma) \cap L^\infty(\Gamma)$. Hence

$$\|\hat{g}\|_{p'} \leq \|T'\| \, \|g\|_{q'} \tag{7}$$

for all such g. Let $FL^{q'}(\Gamma)$ denote the space of functions on G that are Fourier transforms of functions in $L^{q'}(\Gamma)$. Since $L^1(\Gamma) \cap L^\infty(\Gamma)$ is dense in $L^{q'}(\Gamma)$, we conclude from inequality (7) that

$$FL^{q'}(\Gamma) \subset L^{p'}(G). \tag{8}$$

Now, because Γ is not compact, G is not discrete; also, $p' > q = (q')'$. Therefore assertion (a), with the roles of G and Γ interchanged and with p and q replaced by q' and p', states that $FL^{q'}(\Gamma) \not\subset L^{p'}(G)$, contrary to (8). Therefore, $FL^p \not\subset L^q(\Gamma)$ for $2 \leq q < p'$.

As in Theorem 1, we conclude by a Baire-category argument that $FL^p \not\subset \bigcup_{q < p'} L^q(\Gamma)$.

Finally, suppose that Γ is neither compact nor discrete; let $p > 1$. By assertion (a), there exists a function g in $L^p(G)$ such that $\hat{g} \notin \bigcup_{q > p'} L^q(\Gamma)$; by assertion (b), there exists a function h in $L^p(G)$ such that $\hat{h} \notin \bigcup_{q < p'} L^q(\Gamma)$. One of the three functions $\hat{g}$, $\hat{h}$, and $\hat{g} + \hat{h}$ does not lie in $\bigcup_{q \neq p'} L^q(\Gamma)$. This proves assertion (c) and completes the proof of the theorem.

Theorem 2 says that the Hausdorff-Young theorem is the best possible statement about containment of FL^p in $L^q(\Gamma)$. Nevertheless, $FL^p = L^{p'}(\Gamma)$ only if $p = 2$ or Γ is finite (see [11, Vol. II, p. 431] or the corollary to Theorem 3 of the present paper); indeed, there exist interesting spaces that are smaller than $L^{p'}(\Gamma)$ when Γ is infinite and that are known to contain FL^p [12, p. 125, Theorem 3], [18, p. 200, Corollary 3.16], [19, pp. 825-826], [20, Vol. II, p. 121].

Remark 2. In Theorem 2, we used a duality argument to show that (a) implies (b). We can use the same argument to show that Theorem 1 implies the following extension of assertion (b): Suppose that G is a nondiscrete, locally compact, Abelian group, and that E is a subset of G that is not locally null. Then there exists a function f in $L^p(G)$, with $f = 0$ off E, such that $\hat{f} \notin \bigcup_{q<p'} L^q(\Gamma)$. That is,

$$FL^p(E) \not\subset \bigcup_{q<p'} L^q(\Gamma).$$

3. MULTIPLIERS

Let p and q lie in the interval $[1, 2]$. A measurable function ϕ on Γ is called a *multiplier* from $L^p(G)$ to $L^q(G)$ if $\phi \cdot FL^p \subset FL^q$, that is, if $\phi \cdot \hat{f} \in FL^q$ for all f in $L^p(G)$. A more general definition is that, for p and q in $[1, \infty)$, a multiplier from $L^p(G)$ to $L^q(G)$ is a bounded linear operator from $L^p(G)$ to $L^q(G)$ that commutes with translation; however, when $p, q \in [1, 2]$, the equation

$$(Tf)^{\wedge} = \phi \cdot \hat{f}$$

defines a correspondence between such multiplier operators T and the multiplier functions ϕ defined above [9, pp. 102-103], [14, Section 4.1]. Since all locally null functions ϕ correspond to the 0-operator, we follow E. Hewitt and K. A. Ross [11, Vol. I, p. 141] and identify such functions with the 0-function. To indicate how Theorem 1 is related to known results, we use it to give a new proof of a well-known fact about multipliers.

PROPOSITION. *Suppose that* Γ *is not discrete and that* $1 \leq q < p \leq 2$. *Then the only multiplier from* $L^p(G)$ *to* $L^q(G)$ *is the* 0-*function.*

Proof. Suppose that ϕ is a nontrivial multiplier from $L^p(G)$ to $L^q(G)$. Then, for some $\varepsilon > 0$, the set

$$E = \{\gamma \in \Gamma: |\phi(\gamma)| \geq \varepsilon\}$$

is not locally null. Now

$$\phi \cdot FL^p \mid E \subset FL^q \mid E \subset L^{q'}(E).$$

Hence, by the definition of E, we have the relation

$$FL^p \mid E \subset L^{q'}(E),$$

contrary to Theorem 1. Thus the only multipliers from $L^p(G)$ to $L^q(G)$ are trivial.

Using the more general definition of multipliers, one can prove that the proposition actually holds whenever $1 \leq q < p \leq \infty$ [9, p. 99], [14, p. 149].

Remark 3. We now discuss the analogues of Theorems 1 and 2 and of the above proposition for infinite, compact, non-Abelian groups. Fix such a group G with dual $\hat{G}$ [2, p. 75], [11, Vol. II, p. 2]. There exist spaces $\mathscr{L}^p(\hat{G})$ of operator-valued functions on $\hat{G}$ (see [2, p. 144]; these spaces are denoted by $\mathfrak{C}_p(\Sigma)$ in [11, Vol. II, p. 70]) that correspond to the usual ℓ^p-spaces on Γ when G is Abelian and compact. Accordingly, for $1 < p \leq 2$, let $F\mathscr{L}^p$ be the space of functions on G that are inverse transforms of members of $\mathscr{L}^p(\hat{G})$. By R. A. Kunze's extension of the Hausdorff-Young theorem [2, p. 144], [11, Vol. II, p. 229], [13, p. 535],

$$F\mathscr{L}^p \subset L^{p'}(G).$$

Thanks to the referee, we have a proof of Lemma 0 for infinite, compact, non-Abelian groups. Using this result, we can extend Theorem 1 to such groups G and prove that if $E \subset G$ has positive measure and $1 < p \leq 2$, then

$$F\mathscr{L}^p \mid E \not\subset \bigcup_{q > p'} L^q(E).$$

In particular, the analogue of assertion (a) of Theorem 2 holds in this situation. It follows by a duality argument that if G is compact and infinite and $1 \leq p \leq 2$, then

$$FL^p(G) \not\subset \bigcup_{q < p'} \mathscr{L}^q(\hat{G}).$$

Moreover, as in Remark 2, there is a similar non-inclusion result for $FL^p(E)$, provided that $E \subset G$ has positive measure.

Finally, the proposition above can be extended to infinite, compact, non-Abelian groups. The extended result is that if $1 \leq q < p \leq 2$, then the only measurable function ϕ with the property that $\phi \cdot F\mathscr{L}^p \subset F\mathscr{L}^q$ is the 0-function. The usual proof of triviality [14, p. 149] does not work in this situation. A different proof of the extended result appears in [3, p. 360].

4. $\Lambda(q)$-SETS IN NONDISCRETE GROUPS

Fix p in the interval (1, 2). We now consider sets $E \subset \Gamma$ with the property that

$$FL^p \mid E \subset L^q(E) \text{ for some } q < p'. \tag{9}$$

As we noted in the introduction, this inclusion holds for all $q < p'$ if $|E| < \infty$; in particular, if Γ is compact, then every measurable set E satisfies (9). If Γ is not compact and we take $E = \Gamma$, then, by Theorem 2, (9) does not hold. We now show, however, that every noncompact group Γ contains open sets E, of infinite measure, with the property (9).

We use the known fact that every infinite discrete group contains infinite sets E for which (9) holds. Indeed, every such group contains an infinite Sidon set E [16, p. 126] for which $FL^p \mid E \subset \ell^2(E)$ [16, p. 130]. In nondiscrete groups, the analogues of Sidon sets are Helson sets [16, p. 144]. All Helson sets E have property (9); unfortunately, this is so because they all have Haar measure 0 [11, Vol. II, p. 573]. To get sets E that have infinite measure and satisfy (9), we extend to nondiscrete groups another notion of thin set.

Definitions. Let E be a measurable subset of Γ. A function f, in $L^p(G)$ for some $p \leq 2$, is called an *E-function* if $\hat{f}$ is essentially supported by the set E. For $q \in (2, \infty)$, the set E is called a *$\Lambda(q)$-set* if every E-function in $L^2(G)$ is actually in $L^q(G)$.

When Γ is discrete, this definition is equivalent to the usual one [11, Vol. II, p. 420, 37.7 (ii)]. The point is that a set E is a $\Lambda(q)$-set, with $q > 2$, if and only if $FL^{q'} \mid E \subset L^2(E)$ (see [11, Vol. II, p. 421, 37.9 (iv)] for the case where Γ is discrete).

THEOREM 3. *Every noncompact group Γ contains open sets E, of infinite measure, that are $\Lambda(q)$-sets for all q in the interval $(2, \infty)$. These sets have the property that if $1 < p < 2$, then*

$$FL^p \mid E \subset \bigcap_{2 \leq q \leq p'} L^q(E).$$

Proof. By the structure theorem, Γ has an open subgroup of the form $H \times R^n$, where H is compact.

Case 1. Suppose $n > 0$. Normalize the Haar measure on Γ so that the open set $V = H \times (0, 1)^n$ has measure 1. The desired set E will be a union of disjoint translates of V, say

$$E = \bigcup_{j=1}^{\infty} (x_j + V)$$

with the x_j to be chosen later.

We imitate the usual proof that the Rademacher system is a $\Lambda(q)$-set [20, Vol. I, p. 213]. Suppose that f is an E-function in $L^2(G)$ and that $\hat{f}$ is supported by finitely many of the sets $E_j = x_j + V$. Then $\hat{f} \in L^1(\Gamma)$, so that $f \in L^q(G)$ for all q in $[2, \infty)$. Fix an integer $k > 1$; we want to estimate $\|f\|_{2k}$. Now

$$\int_G |f(x)|^{2k} dx = \int_G |f(x)^k|^2 dx = \int_\Gamma |(f^k)^\wedge(\gamma)|^2 d\gamma .$$

Note that the transform $(f^k)^\wedge$ of f^k is the convolution product

$$\hat{f}^{*k} = \hat{f} * \hat{f} * \cdots * \hat{f}$$

of k copies of $\hat{f}$.

Let f_j be the function on Γ that is equal to $\hat{f}$ on the set E_j and vanishes elsewhere. To expand

$$\hat{f}^{*k} = \left(\sum_{j=1}^{\infty} f_j \right)^{*k} ,$$

we use multiindex notation. The symbol α denotes a sequence $\{\alpha_j\}_{j=1}^{\infty}$ of nonnegative integers with all but finitely many terms equal to 0. Let $|\alpha| = \sum_{j=1}^{\infty} \alpha_j$ and $\alpha! = \prod_{j=1}^{\infty} \alpha_j!$. Denote by f_α the finite convolution product $f_1^{*\alpha_1} * f_2^{*\alpha_2} * \cdots$. Then

$$\hat{f}^{*k} = \sum_{|\alpha| = k} \frac{k!}{\alpha!} f_\alpha .$$

Now the support of f_α is contained in the set $E_\alpha = \sum_{\alpha_j \neq 0} \alpha_j E_j$, where $\alpha_j E_j$ denotes the sum $E_j + E_j + \cdots + E_j$ of α_j copies of E_j. Suppose for the moment that, whenever α and β are distinct multiindices with $|\alpha| = |\beta| = k$, the sets E_α and E_β are disjoint. Then

$$\int_\Gamma |\hat{f}^{*k}(\gamma)|^2 d\gamma = \sum_{|\alpha| = k} \left(\frac{k!}{\alpha!} \right)^2 \int_\Gamma |f_\alpha(\gamma)|^2 d\gamma .$$

Let $c_j = \|f_j\|_2$, and let $c^\alpha = \prod_{\alpha_j \neq 0} c_j^{\alpha_j}$. Because $|V| = 1$, we have the inequalities $\|f_j\|_1 \leq c_j$ and $\|f_\alpha\|_2 \leq c^\alpha$. Therefore

$$\int_\Gamma |f(x)|^{2k} dx \leq \sum_{|\alpha| = k} \left(\frac{k!}{\alpha!} \right)^2 c^{2\alpha} .$$

Compare this expression with

$$\left(\int_G |f(x)|^2\,dx\right)^k = \left(\sum_{j=1}^{\infty} c_j^2\right)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\, c^{2\alpha}.$$

Clearly,

$$(\|f\|_{2k})^{2k} \leq k!\,(\|f\|_2)^{2k}. \tag{10}$$

Since the set of E-functions f in $L^2(G)$ with $\hat{f}$ supported by finitely many of the sets E_j is dense in the set of all E-functions in $L^2(G)$, we conclude that inequality (10) persists for all E-functions f. Therefore E is a $\Lambda(2k)$-set.

Now define x_j in $H \times R^n$ to be $(0, 4^j, 0, 0, \cdots)$. For large, fixed k, the sets E_α, for distinct α with $|\alpha| = k$, need not be disjoint, but it is possible to split $\{x_j\}_{j=1}^{\infty}$ into a finite number of subsequences, depending on k, so that, for each subsequence, the corresponding sets E_α with $|\alpha| = k$ are disjoint for distinct α. This makes E a finite union of $\Lambda(2k)$-sets; as in [15, p. 217], E must be a $\Lambda(2k)$-set. Finally, since E is of type $\Lambda(2k)$ for all integers $k > 1$, it is of type $\Lambda(q)$ for all q in the interval $(2, \infty)$.

Case 2. Suppose $n = 0$. If the open subgroup H is finite, then Γ is discrete and infinite; as we noted above, Γ then contains infinite Sidon sets E that are of type $\Lambda(q)$ for all $q < \infty$ [11, Vol. II, p. 423].

If the subgroup H is infinite, normalize the Haar measure on Γ so that $|H| = 1$. Let E be a union of distinct cosets of H, say

$$E = \bigcup_{j=1}^{\infty} (x_j + H) = \bigcup_{j=1}^{\infty} E_j.$$

As in Case 1, it is sufficient to prove that E is a $\Lambda(2k)$-set for all integers $k > 1$. Suppose that f is an E-function in $L^2(G)$ with $\hat{f}$ supported by finitely many of the cosets E_j.

We keep the notation of Case 1 and let $g \in L^2(G)$ satisfy the condition that $\hat{g} \equiv c_j$ on each set E_j and $\hat{g} = 0$ otherwise; then $\|g\|_2 = \|f\|_2$. If $|\alpha| = k > 1$, then each function f_α is supported by the coset E_α; also, $\|f_\alpha\|_\infty \leq c^\alpha$. On the other hand, the corresponding function g_α is identically equal to c^α on the coset E_α, and $g_\alpha \equiv 0$ otherwise. Therefore

$$|\hat{f}^{*k}(\gamma)| \leq \sum_{|\alpha|=k} \frac{k!}{\alpha!}\,|f_\alpha(\gamma)| \leq \sum_{|\alpha|=k} \frac{k!}{\alpha!}\, g_\alpha(\gamma) = \hat{g}^{*k}(\gamma),$$

for all $\gamma \in \Gamma$. We conclude that $\|f\|_{2k} \leq \|g\|_{2k}$.

Now, because $\hat{g}$ is constant on cosets of H, the function g is essentially supported by $H^\perp$, the annihilator of H [16, p. 53]. Note that $H^\perp$ is a compact open subgroup of G, because H is open and compact in Γ; also, applying the Plancherel theorem to the characteristic function of H, we see that the Haar measure in G of $H^\perp$ is 1. Therefore $\|g\|_q$ is the same whether we view g as a function on all of G and take the norm in $L^q(G)$, or restrict g to $H^\perp$ and take the norm with respect to the Haar measure in the compact group $H^\perp$.

The dual group of $H^{\perp}$ is Γ/H [16, p. 35], and Γ/H is infinite because H is compact and Γ is not compact. Choose the sequence $\{x_j\}_{j=1}^{\infty}$ in Γ so that the cosets $x_j + H$ are distinct and form an infinite Sidon set in Γ/H. Now view g as a function on $H^{\perp}$; it is easy to verify that $\hat{g}$ is supported by the Sidon set $\{x_j + H\}_{j=1}^{\infty}$ in Γ/H. Therefore $\|g\|_{2k} \leq C\sqrt{k}\,\|g\|_2$ [16, p. 130]. Hence

$$\|f\|_{2k} \leq \|g\|_{2k} \leq C\sqrt{k}\,\|g\|_2 = C\sqrt{k}\,\|f\|_2 \tag{11}$$

for all E-functions f that have $\hat{f}$ supported by finitely many of the cosets E_j; as in Case 1, inequality (11) persists for all E-functions f. Therefore E is a $\Lambda(2k)$-set for all integers $k > 1$ and a $\Lambda(q)$-set for all $q > 2$.

We have shown that every noncompact group Γ contains an open set, of infinite measure, that is of type $\Lambda(q)$ for all $q > 2$. Consider such a set E, and let $1 < p < 2$. Then the Plancherel transform $T: L^2(E) \to L^2(G)$ actually takes $L^2(E)$ into $L^{p'}(G)$. Therefore the dual mapping T' takes $L^p(G)$ into $L^2(E)$; as in the proof of Theorem 2, we see that $T'f = \hat{f} \mid E$ for all f in $L^1(G) \cap L^2(G)$, hence for all f in $L^p(G)$. Thus $FL^p \mid E \subset L^2(E)$ and

$$FL^p \mid E \subset L^2(E) \cap L^{p'}(E) \subset L^q(E)$$

for all q in the interval $[2, p']$. This completes the proof of the theorem.

COROLLARY [10, p. 572], [11, Vol. II, p. 431]. *If* G *is infinite and* $1 < p < 2$, *then* $FL^p \neq L^{p'}(\Gamma)$.

Proof. Suppose first that Γ is not compact; let $E \subset \Gamma$ be an open $\Lambda(p')$-set of infinite measure. Then

$$FL^p \mid E \subset L^2(E)$$

but

$$L^{p'}(\Gamma) \mid E = L^{p'}(E) \not\subset L^2(E).$$

Therefore $FL^p \neq L^{p'}(\Gamma)$.

Now, when G is infinite, either G or Γ is not compact. Therefore at least one of the two mappings

$$\hat{}: L^p(G) \to L^{p'}(\Gamma) \quad \text{and} \quad \hat{}: L^p(\Gamma) \to L^{p'}(G)$$

is not surjective. But the maps are dual and injective, and it follows that if one of them is surjective, then the other is also surjective [11, Vol. II, p. 713, (E.9)]. Hence

$$FL^p(G) \neq L^{p'}(\Gamma) \quad \text{and} \quad FL^p(\Gamma) \neq L^{p'}(G).$$

Remark 4. It follows from the definition that for $q > 2$, a $\Lambda(q)$-set E has the further property that, if $p < 2$, then every E-function in $L^p(G)$ is also in $L^q(G)$; to see this, one argues as in [15, pp. 204-205]. When Γ is discrete, it follows trivially from our definition of $\Lambda(q)$-sets that for $2 < p < q$ every E-function in $L^p(G)$ is in $L^q(G)$; when Γ is not discrete, one can extend the definition of E-functions to the case $p > 2$ (see [8, pp. 478-479] and [7, p. 469], or [9, pp. 143 and 150]); but it is not clear whether the $\Lambda(q)$-sets E defined above have the property that for $2 < p < q$, every E-function in $L^p(G)$ is in $L^q(G)$. Also, in the case where Γ is not discrete, it is possible for an E-function to belong to $L^2(G)$ but not to $L^p(E)$ for any $p < 2$; to obtain such a function, we simply take the inverse Plancherel transform of a function on Γ that is supported by E and belongs to $L^2(E)$ but not to $L^q(E)$ for $q > 2$. Finally, we note that the assumption $p > 1$ in Theorem 3 is essential; for FL^1 contains functions in $C_0(\Gamma)$ that tend to 0 arbitrarily slowly [11, Vol. II, p. 286, 32.47 (b)].

Remark 5. The analogue of Theorem 3 for an infinite, compact, non-Abelian group G is that the dual $\hat{G}$ contains an infinite set that is of type $\Lambda(p)$ for all finite p. This statement is false for some groups G [11, Vol. II, p. 434, 37.21(b)]. On the other hand, the analogue of the Corollary to Theorem 3 does hold for compact, non-Abelian groups G [2, p. 148].

Remark 6. We end the paper with a variation on its main theme: For what indices p and q and what sets $E \subset \Gamma$ is $FL^p \mid E$ equal to $L^q(E)$? To avoid trivialities, suppose that E is infinite and not locally null.

Of course, $FL^2 \mid E = L^2(E)$. On the other hand, $FL^1 \mid E$ is never equal to $L^q(E)$ for any q. Indeed, when Γ is discrete, FL^1 contains elements of $C_0(\Gamma)$ that tend to 0 arbitrarily slowly [11, Vol. II, p. 286, 32.47(b)]. When Γ is not discrete, the compact sets E satisfying the condition $C(E) \subset FL^1 \mid E$ are Helson sets and have measure 0 [11, Vol. II, p. 573]; therefore $FL^1 \mid E = L^q(E)$ for no compact set E of positive measure, hence for no set E that is not locally null.

There remains the case $1 < p < 2$. Since E is infinite and not locally null, the relation $FL^p \mid E = L^q(E)$ can hold for at most one index q. Suppose that Γ is discrete. Then $FL^p \mid E = \ell^2(E)$ if and only if E is a $\Lambda(p')$-set [11, Vol. II, p. 421]. Therefore, if E is not a $\Lambda(p')$-set then $FL^p \mid E \neq \ell^2(E)$; but even in this case, E contains an infinite $\Lambda(p')$-set, E' say, and we see that $FL^p \mid E' = \ell^2(E')$, while, if $q \neq 2$,

$$\ell^q(E) \mid E' = \ell^q(E') \neq \ell^2(E').$$

We conclude that $FL^p \mid E = \ell^q(E)$ only if $q = 2$ and E is a $\Lambda(p')$-set.

Finally, suppose that Γ is not discrete and that $1 < p < 2$. Then $FL^p \mid E = L^q(E)$ for no index q. For suppose that $FL^p \mid E = L^q(E)$ for some set E that is not locally null. Replacing E by a subset, if necessary, we can assume that E is compact and has positive measure. Now, by the Hausdorff-Young theorem,

$$L^q(E) = FL^p \mid E \subset L^{p'}(E);$$

because Γ is not discrete, this implies that $q \geq p'$. But by Theorem 1, the condition $FL^p \mid E = L^q(E)$ implies that $q \leq p'$. Hence $q = p'$, and the mapping

$$\hat{}: L^p(G) \to L^{p'}(E)$$

is surjective. Therefore the dual mapping

$$\check{}: L^p(E) \to L^{p'}(G)$$

has closed range [11, Vol. II, p. 713, E.9]. Consider, however, the functions Φ, defined for $\varepsilon = 1/2$ and $E = M$ in [6, p. 182]. They satisfy the condition $\|\Phi\|_p \geq (|E|/2)^{1/p}$, because $\Phi \equiv 1$ on at least half of the set E. On the other hand,

$$\|\check{\Phi}\|_{p'} = O(N^{-1+2/p'}) \to 0 \quad \text{as } N \to \infty.$$

This shows that the one-to-one mapping

$$\hat{}: L^p(E) \to L^{p'}(G)$$

does not have closed range. Therefore $FL^p \mid E = L^q(E)$ for no q.

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