THE SINGULAR SET OF A BOUNDED ANALYTIC FUNCTION

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INTRODUCTION

A bounded holomorphic function S on the open unit disc Δ is a *singular function* if it has the form

$$S(z) = \exp\left(- \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) \right) \qquad (z \in \Delta),$$

where μ is a real, nonnegative measure that is singular with respect to Lebesgue measure. The measure μ is called the *measure associated with* S, while S is called the *singular function determined by* μ. The function S is called *trivial* if $\mu = 0$. Each bounded holomorphic function g on Δ has a canonical factorization $g = Sh$, where S is the greatest singular function (possibly trivial) that divides g, and where h is bounded and holomorphic on Δ.

Let f be a bounded holomorphic function on Δ. J. G. Caughran and A. L. Shields [3] have raised the question how many complex numbers c there are with the property that $f(z) - c$ has a nontrivial singular factor. The set of such c may be rather large; for example, if K is a compact set of logarithmic capacity 0 in Δ, then there exists a bounded holomorphic function f such that $f(z) - c$ has a nontrivial singular factor for each $c \in K$ (see [3]). If, however, we ask how many numbers c there are for which $f(z) - c$ has a singular factor whose associated measure has positive mass at one or more points, then the size of the set of such values c is substantially diminished; Theorem 1 asserts that the set can be at most countable. Indeed, we show more. We shall say that a point λ ($|\lambda| = 1$) is in the *singular set* of f if there exists a number c such that $f(z) - c$ has a singular factor whose associated measure has positive mass at λ. Theorem 1 asserts that the singular set of f is countable. If λ is in the singular set of f, then $f(z) \to c$ as $z \to \lambda$ nontangentially; it follows that the set of such numbers c is countable. Theorem 2 gives a restriction on the weights that can occur in the associated measures. Theorem 3 shows that the countability restriction in Theorem 1 can not be improved. Theorems 1, 2, and 3 are in Section 2; Section 1 contains two results on angular derivatives that we need in the proofs of Theorems 1 and 2.

1. SINGULAR FACTORS AND ANGULAR DERIVATIVES

Definition. Let ϕ be holomorphic in Δ and bounded by 1, and suppose that $\phi(z) \to \beta$ ($|\beta| = 1$) as z approaches $e^{i\theta}$ nontangentially within Δ. The function ϕ has an *angular derivative* with value γ at $e^{i\theta}$ if

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$$\lim \frac{\beta - \phi(z)}{e^{i\theta} - z} = \gamma \qquad \text{as } z \to e^{i\theta} \text{ nontangentially}.$$

This is equivalent to the condition that $\lim \phi'(z) = \gamma$ as $z \to e^{i\theta}$ nontangentially (see [1, Section 299]).

PROPOSITION 1. *Let* ϕ *be holomorphic and bounded by* 1 *in* Δ, *and suppose that* $\phi(z) \to \beta$ ($|\beta| = 1$) *as* $z \to e^{i\theta}$ *nontangentially. Let* S *be the singular function determined by a mass of weight* 1 *at* $z = \beta$. *Then* $S(\phi(z))$ *has a singular factor whose associated measure has positive mass at* $e^{i\theta}$ *if and only if* ϕ *has an angular derivative at* $e^{i\theta}$. *The absolute value of the angular derivative of* ϕ *at* $e^{i\theta}$ *is the reciprocal of the weight at* $e^{i\theta}$ *of the measure associated with the singular factor of* $S(\phi(z))$.

Proof. The identity $|S(z)| = \exp(-(1 - |z|^2)(|\beta - z|)^{-2})$ implies that

$$|S(\phi(z))| = \exp(-(1 - |\phi(z)|^2)(|\beta - \phi(z)|)^{-2}).$$

Now ϕ has an angular derivative with absolute value no more than $\delta > 0$ at $e^{i\theta}$ if and only if

$$(|\beta - \phi(z)|^2)(1 - |z|^2) \leq \delta(|e^{i\theta} - z|^2)(1 - |\phi(z)|^2) \qquad (|z| < 1)$$

(see [1, Sections 296-298]). $S(\phi(z))$ is divisible by a singular function whose associated measure has mass ε at $e^{i\theta}$ if and only if

$$|S(\phi(z))| \leq \exp(-\varepsilon(1 - |z|^2)(|e^{i\theta} - z|)^{-2}).$$

This inequality and the inequality above, combined with the formula for $|S(\phi(z))|$, yield the conclusions of the proposition.

We shall need the following result of S. Warschawski on the existence of the angular derivative (see [5, Chapter 9, Theorem 9, p. 366]).

PROPOSITION 2. *Let* R *be a Jordan domain in the upper half-plane* U *that is tangent to the real axis at the origin. Suppose* $R \cup \{0\}$ *contains the graph of a function* h *defined on* $[-\delta, \delta]$, *where* $y = h(x)$ *for* $-\delta < x < \delta$, *and where* h *is even, positive, continuous, and increasing on* $(0, \delta]$ *and*

$$\int_0^\delta x^{-2} h(x)\, dx < \infty.$$

Let ϕ *be a one-to-one conformal mapping of* U *onto* R *with* $\phi(s) = 0$ *for some real number* s. *Then there is a number* γ ($0 < \gamma < \infty$) *such that*

$$\lim \phi'(w) = \gamma \qquad \text{as } w \to s \text{ nontangentially, with } w \in U.$$

2. COUNTABILITY OF THE SINGULAR SET

THEOREM 1. *The singular set of a bounded holomorphic function is countable.*

Proof. Although the theorem concerns functions holomorphic on Δ, we begin by making several estimates and a construction in the upper half-plane U, since the technicalities are less formidable there. Let $\tau(z) = (z - i)(z + i)^{-1}$; then τ is a one-to-one conformal mapping of U onto Δ. Let t be a real number; we shall denote by R(t) the region in U that is symmetric about the line $\Re z = t$, bounded on the right by the graph of $y = (x - t)^{3/2}$, and bounded above by the line $\Im z = 1$. Let $\lambda = \tau(t)$, and let S be the singular function on Δ determined by a mass of weight ε at $z = \lambda$. Then the function

$$S_1(z) = S(\tau(z)) = \exp\left(i\varepsilon \frac{1+tz}{t-z}\right)$$

is holomorphic and bounded by 1 on U, and

$$|S_1(z)| = \exp(-\varepsilon(1+t^2)y|z-t|^{-2}) \qquad (z = x+iy \in U).$$

In particular, for $z \in R(t)$, we have the inequality

$$|S_1(z)| \leq \exp\left(-\frac{1}{2}\varepsilon(1+t^2)y^{-1/3}\right).$$

Furthermore,

$$|S_1'(z)| = \varepsilon(1+t^2)(|z-t|)^{-2}|S_1(z)|,$$

and hence $S_1'(z) \to 0$ as $z \to t$ ($z \in R(t)$), since S_1 goes to zero exponentially in R(t).

Let f be a bounded holomorphic function on Δ, and let $f_1(z) = f(\tau(z))$. Let λ be a point of the singular set of f, let $c = \lim_{r \to 1} f(r\lambda)$, and let $\tau(t) = \lambda$. Because λ is in the singular set of f, we have the factorization

$$f(z) - c = S(z;\lambda)g(z) \qquad (z \in \Delta),$$

where $S(z;\lambda)$ is the singular function on Δ determined by a mass of weight ε at λ (ε depends on λ), and where g is bounded and holomorphic on Δ. Thus

$$f_1(z) - c = S(\tau(z);\lambda)g(\tau(z)).$$

This implies that

$$f_1'(z) = \tau'(z)S'(\tau(z);\lambda)g(\tau(z)) + \tau'(z)g'(\tau(z))S(\tau(z);\lambda).$$

The first term on the right-hand side goes to zero as $z \to t$ with $z \in R(t)$, by our comments above. The second term also goes to zero as $z \to t$ with $z \in R(t)$, because $g'(\tau(z))$ is no larger than a constant times y^{-1}. Hence, $f_1'(z) \to 0$ as $z \to t$ with $z \in R(t)$.

Let E_n consist of those real numbers s in the interval $[-n, n]$ for which $|f_1'(z)| \leq 1$ when $z \in R(s)$ and $0 < \Im z < 1/n$. Note that E_n is a closed set; moreover, if λ lies in the singular set of f and $\tau(t) = \lambda$, then t lies in E_n for some n. We shall show that each E_n is countable, and this will prove the theorem. For the remainder of the proof, we shall work with one set E_n; therefore, to simplify the notation, we denote E_n by E.

Let V be the union of the rectangle

$$\{x+iy\colon |x| < n+1 \text{ and } n^{-1} < y < 2\}$$

and the set $\bigcup\{R(t)\colon t \in E\}$. The set V is open, and it is bounded by a rectifiable

Jordan curve. Furthermore, f_1' is bounded on V. Finally, if ψ is a one-to-one conformal mapping of U onto V and $\psi(s) = t \in E$, then Proposition 2 assures us that ψ has an angular derivative at s. Let $\Omega = \tau(V)$; the domain Ω is bounded by a rectifiable simple closed curve, and f' is bounded on Ω. Select a point $b \in \Omega$, and let ϕ be the Riemann mapping of Δ onto Ω with $\phi(0) = b$. Let $F = \tau(E)$; then ϕ has an angular derivative at each point of $G = \phi^{-1}(F)$, since ψ has an angular derivative at each point of $\psi^{-1}(E)$.

Let $g(z) = f(\phi(z))$. Then $g' = (f' \circ \phi)\phi'$, and because the boundary of Ω is rectifiable, ϕ' is in the Hardy space $H^1(\Delta)$ ([4, Theorem 3.12]). Since f' is bounded on Ω, g' is also in $H^1(\Delta)$. Consequently, by a theorem of Caughran [2, Theorem 1], the singular factor of g divides g'; similarly, the singular factor of g - c divides (g - c)' = g'. By Proposition 1, the measure associated with the singular factor of g - c has mass at some point of G, and as c varies, each point of G must therefore appear. Hence, the measure associated with the singular factor of g' has mass at every point of G. Thus G is countable, and therefore F and E are also countable. This completes the proof.

Remark. Let K be a compact set in Δ, of logarithmic capacity 0, and let f be an inner function on Δ whose range is Δ - K. Then, for each $c \in K$, $f_c = (f - c)(1 - \bar{c}f)^{-1}$ must be a singular function. Theorem 1 then implies that for all but a countable number of points c in K the measure associated with f_c must be continuous (no point masses). Just how the measure associated with f_c changes as c varies in K is unknown.

THEOREM 2. *Let* $\delta > 0$. *Then there are only a finite number of points in the singular set of* f *at which the measure associated with the singular factor of* f(z) - c *has mass exceeding* δ.

Proof. The proof is a continuation of that of Theorem 1, with the same notation. We first note that corresponding to δ there is an n such that t lies in E_n for all those points t for which the singular factor of f(z) - c has mass exceeding δ at $\lambda = \tau(t)$. This follows immediately from the estimates on the size of $|f_1'|$ in the proof of Theorem 1. We shall work with this E_n and again drop the subscript. The remaining step in the proof is to show that the weight of the measure associated with the singular factor of g - c exceeds a constant multiple of the corresponding weight for the singular factor of f - c. This will follow from the final assertion of Proposition 1, if we show that the absolute value of the angular derivative of ϕ is bounded above on G.

Let $e^{i\theta} \in G$, $\lambda = \phi(e^{i\theta})$, $t = \tau^{-1}(e^{i\theta})$, and let $\omega = \tau(R(t))$. Then ω is a subset of Ω and is bounded by a Jordan curve. Let π be a one-to-one conformal mapping of Δ onto ω with $\pi(0) = b$ (there is no loss in assuming $b \in \omega$). Let $\rho = \phi^{-1} \circ \pi$; then ρ maps Δ into Δ, $\rho(0) = 0$, and if $e^{is} = \pi^{-1}(\lambda)$, then $\rho(e^{is}) = e^{i\theta}$. The function ρ must have an angular derivative at e^{is}; to see this, we observe that both ϕ and π have angular derivatives at e^{is} and $e^{i\theta}$, respectively, by Proposition 2, because, with the obvious notation,

$$\phi'(\rho(e^{is}))\rho'(e^{is}) = \pi'(e^{is}).$$

However, $|\rho'(e^{is})| \geq 1$ (see [1, Section 301]), so that $|\phi'(e^{i\theta})| \leq |\pi'(e^{is})|$; this establishes the desired inequality.

Theorem 3 below shows that the countability conclusion of Theorem 1 can not be improved. For part (ii), we recall that a Carleson set is a closed set of Lebesgue measure zero in the unit circle whose complementary intervals have length ε_n, where $-\sum \varepsilon_n \log \varepsilon_n$ is finite.

THEOREM 3. (i) *Let* $E = \{\lambda_i\}$ *be a countable set in the unit circle* T, *and let* $\{\varepsilon_i\}$ *be a summable sequence of positive numbers. Then there exists a function* f, *continuous on* $\Delta \cup T$ *and holomorphic on* Δ, *such that* $f(z) - f(\lambda_i)$ *is divisible by the singular function determined by a mass of weight* ε_i *at* λ_i $(i = 1, 2, \cdots)$.

(ii) *If* E *is closed but is not a Carleson set, then* f *assumes infinitely many distinct values on* E.

Proof. (i) Let μ be the positive measure with mass ε_i at λ_i $(i = 1, 2, \cdots)$, and let I be the singular function determined by μ. Let

$$f(z) = \int_0^z I(w)\,dw \qquad (z \in \Delta).$$

Then f has a bounded derivative, and thus it is continuous on the closed unit disc. A straightforward computation (or Theorem 2 of [2]) shows that $f(z) - f(\lambda_i)$ is divisible by the singular function determined by a mass of weight ε_i at λ_i.

(ii) It is known that a closed set E is a Carleson set if and only if there exists a function g that is analytic on Δ and continuous on $\Delta \cup T$ and belongs to the class Lip 1 on T, with $g = 0$ precisely on E (see [2, Theorem A]). In (i), the function f is in Lip 1 on T. Suppose f assumes only the values $c_1, \cdots, c_N$ on E. Let $g_i = f - c_i$ for $i = 1, \cdots, N$; then g_i vanishes on a certain closed set E_i in E, where $E_1 \cup \cdots \cup E_N = E$, and E_i is therefore a Carleson set. Since the union of finitely many Carleson sets is a Carleson set, this implies that E is also a Carleson set, a contradiction.

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