# ALMOST SUFFICIENTLY LARGE SEIFERT FIBER SPACES

Wolfgang Heil

All the known examples of 3-manifolds that are not sufficiently large but are almost sufficiently large are Seifert fiber spaces [14], [7]. The first explicit example of this kind was constructed by H. Boehme [1]. In [14, p. 87], F. Waldhausen conjectured that in fact all (orientable) irreducible Seifert fiber spaces with infinite fundamental group are almost sufficiently large. We prove this in Section 4. In Sections 2 and 3, we list the Seifert fiber spaces that are not $P^2$-irreducible and not sufficiently large. In Section 5, we describe the Seifert fiber spaces that contain incompressible surfaces of negative Euler characteristic.

## 1. PRELIMINARIES

Let M be a 3-manifold, and let F denote a 2-sided surface embedded in M, with $F \neq S^2$, $F \neq P^2$, and $F \cap \partial M = \partial F$. Then F is *incompressible* in M provided the mapping $i_*: \pi_1(F) \to \pi_1(M)$ induced by inclusion is injective. The manifold M is *sufficiently large* if M contains a 2-sided incompressible surface F ($F \neq S^2$, $P^2$), and M is *almost sufficiently large* if there exists a finite cover $\tilde{M}$ of M that is sufficiently large. We say that M is $P^2$-*irreducible* if M contains no 2-sided projective planes and is irreducible (that is, if each 2-sphere in M bounds a 3-cell). $M_1 \# M_2$ denotes a connected sum of the two 3-manifolds $M_1$ and $M_2$.

For a discussion of Seifert fiber spaces, see [10].

## 2. NONIRREDUCIBLE SEIFERT FIBER SPACES

We denote by $\hat{M}$ the manifold obtained from M by capping off the 2-spheres of $\partial M$ with 3-balls.

PROPOSITION 1. *Let* M *be a compact* 3-*manifold such that* $\partial M$ *contains no projective planes. Suppose the universal cover of* M *embeds in* $S^3$. *If* $\pi_1(M)$ *contains a nontrivial cyclic normal subgroup, and if* $\hat{M}$ *is not* $P^2$-*irreducible, then either*

(a) $\hat{M}$ *is an* $S^2$-*bundle over* $S^1$,

(b) $\hat{M} \approx P^2 \times S^1$, *or*

(c) $\hat{M} \approx P^3 \# P^3$ (*a connected sum of two projective spaces*).

*Proof.* (i) Suppose $\hat{M}$ is not irreducible. Then $\hat{M}$ contains an essential 2-sphere $S^2$. If $S^2$ separates $\hat{M}$, then $\pi_1(\hat{M})$ is a nontrivial free product, and since $\pi_1(\hat{M})$ has a cyclic normal subgroup, $\pi_1(\hat{M}) \approx Z_2 * Z_2$ (see [10, p. 228]). Thus $H_1(M)$ is finite, and therefore $\hat{M}$ is closed and orientable (since $\partial\hat{M}$ contains no $P^2$ and no

Received December 9, 1972.
This research was partially supported by NSF Grant GP-19964.

Michigan Math. J. 20 (1973).

$S^2$). By Kneser's conjecture [11], $\hat{M} \approx M_1 \# M_2$, where $\pi_1(M_i) \approx Z_2$. But $M_i$ is irreducible (since $\pi_1(M_i)$ is not infinite cyclic and is not a free product), and its universal cover embeds in $S^3$. Therefore, by [8], $M_i \approx P^3$ (i = 1, 2), and we get case (c) of the proposition.

Now assume that no essential 2-sphere in $\hat{M}$ separates. Then, since $\hat{M}$ is not a nontrivial connected sum, $\hat{M}$ must be an $S^2$-bundle over $S^1$.

(ii) Suppose $\hat{M}$ is irreducible, and suppose it contains a 2-sided projective plane $P^2$. Let $p: M' \to \hat{M}$ be the 2-fold orientable cover. Then $\partial M'$ contains no $S^2$, and hence $\hat{M}' = M'$. The group $\pi_1(\hat{M})$ is infinite; otherwise $\pi_1(\hat{M}) \approx Z_2$, and $\partial\hat{M}$ contains two projective planes [2].

If N denotes the cyclic normal subgroup of $\pi_1(\hat{M})$, the group $N' = p_*^{-1}(N)$ is a cyclic normal subgroup of $\pi_1(M')$. If $N' = 1$, then $N = Z_2$, and since N is normal, it is central in $\pi_1(\hat{M})$. If t is an element of infinite order of $\pi_1(\hat{M})$, then $\mathbb{Z}(t) \times \mathbb{Z}_2$ is a subgroup of $\pi_1(\hat{M})$. From [2, Theorem 9.5] and the fact that $\hat{M}$ is irreducible, we deduce that $\pi_1(\hat{M}) = \mathbb{Z}(t) \times \mathbb{Z}_2$. It follows that $\pi_1(M') = \mathbb{Z}(t)$, and that $\hat{M}' \approx M_1' \# \Sigma$, where $M_1'$ is prime and $\Sigma$ is a homotopy 3-sphere. Since the universal cover of $M'$ embeds in $S^3$, we see that $M' \approx \hat{M}' \approx M_1' \approx S^2 \times S^1$. Therefore, $P^2$ lifts to a nonseparating $S^2$ in $S^2 \times S^1$, and it follows from [9] that $M \approx P^2 \times S^1$.

If $N' \neq 1$, we can apply case (i) to $M'$, because a 2-sided $P^2$ in $\hat{M}$ lifts to an essential 2-sphere $S^2$ in $M'$. The case where $M' \approx S^1 \times S^2$ has been discussed above; thus we assume $M' \approx P^3 \# P^3$. The projective plane $P^2$ of M lifts to a sphere that separates the two summands of $M'$. Since $P^2$ is 2-sided in M, the covering translation could not interchange the two summands $P^3$ of $M'$, and there would exist a 3-manifold whose boundary is a $P^2$, which is impossible.

The following Corollary was proved implicitly by F. Waldhausen [12] for the case where the manifold M is orientable. If M is nonorientable we can also prove it, by examining the invariants of the orientable double covers, as described by Seifert [10].

COROLLARY. *Let* M *be a Seifert fiber space.*

(a) M *is not irreducible if and only if* M *is either an* $S^2$-*bundle over* $S^1$ *or the manifold* $P^3 \# P^3$.

(b) M *contains a* 2-*sided projective plane* $P^2$ *if and only if* $M \approx P^2 \times S^1$.

*Proof.* If $M \neq S^3$, then a fiber of M generates a nontrivial cyclic normal subgroup. In order to apply the proposition, we have to show that the universal cover $\tilde{M}$ of M embeds in $S^3$. First assume M is closed. We can assume that no multiple of a fiber of M lifts to a closed curve in $\tilde{M}$, for otherwise it follows from [10, Sätze 19 and 11] that $\tilde{M} \approx S^3$. Let $q: M \to f$ be the projection of M onto its orbit surface f. Let $M_*$ be the $S^1$-bundle over the surface $f_*$, obtained by drilling out fibered neighborhoods of the exceptional fibers and disk neighborhoods of the exceptional points from M and f, respectively (see [10]). Let $\tilde{f}$ be the universal cover of f, and let $\tilde{f}_*$ be the part of $\tilde{f}$ that lies over $f_*$. Then $\tilde{M}_* \approx \tilde{f}_* \times R^1$ covers $M_*$, and $\tilde{M}$ is obtained from $\tilde{M}_*$ by filling in copies of $D^2 \times R^1$ (that cover the drilled-out exceptional fibers of M) along $\partial D^2 \times R^1$. Hence $\tilde{M} \approx \tilde{f} \times R^1$. If $f \approx S^2$ or $f \approx P^2$, then $\tilde{M} \approx S^2 \times R^1 \subset S^3$. In every other case, $\tilde{M} \approx R^2 \times R^1 \subset S^3$.

If M is not closed, we can assume that M is not a solid torus and not a solid Klein bottle. Therefore, $\partial M$ is incompressible (see the proof of Proposition 2 below), and if D(M) is the double of M (obtained by identifying two copies of M along boundary components), an inclusion i: $M \to D(M)$ induces an injection $i_*: \pi_1(M) \to \pi_1(D(M))$. Therefore $\tilde{M}$ is a submanifold of the universal cover $\tilde{D}(M)$ of D(M), which embeds in $S^3$, by the first part of the proof.

## 3. SUFFICIENTLY LARGE SEIFERT FIBER SPACES

A Seifert fiber space is uniquely determined by its "class" and a system of invariants [10]. For example, the notation

$$M = (O\,o;\, 0 \mid b;\, \alpha_1, \beta_1;\, \cdots;\, \alpha_r, \beta_r)$$

means that M is orientable, that the orbit surface is $S^2$, and that M has r exceptional fibers of orders $\alpha_1, \cdots, \alpha_r$ with fibered neighborhoods of type $V(\alpha_1, \beta_1), \cdots, V(\alpha_r, \beta_r)$, respectively. The invariant b specifies the type of the $S^1$-bundle one obtains by drilling out the exceptional fibers and replacing them by ordinary fibers.

PROPOSITION 2. *A Seifert fiber space* M *is sufficiently large, except in the following cases:*

(1) M *is a lens space.*

(2) M *is an* $S^2$*-bundle or* $P^2$*-bundle over* $S^1$.

(3) $M \approx P^3 \# P^3$.

(4) $M = (O\,o;\, 0 \mid b;\, \alpha_1, \beta_1;\, \alpha_2, \beta_2;\, \alpha_3, \beta_3)$, *where*

$$b\,\alpha_1 \alpha_2 \alpha_3 + \beta_1 \alpha_2 \alpha_3 + \beta_2 \alpha_1 \alpha_3 + \beta_3 \alpha_1 \alpha_2 \neq 0.$$

*Proof.* By Proposition 1, we can assume that M is $P^2$-irreducible (otherwise, we get case (2) or (3)). If $\partial M \neq \emptyset$, or M is nonorientable, or $H_1(M)$ is infinite, then M is sufficiently large [14], [4]. Therefore assume that M is closed and orientable and that $H_1(M)$ is finite. Since $p_*: H_1(M) \to H_1(f)$ is surjective, it follows that the orbit surface f is $S^2$ or $P^2$. If $f \approx P^2$, and if in addition M has at least two exceptional fibers, let D be a disk on f containing all the exceptional points of f. Then $p^{-1}(\partial D)$ is an incompressible torus in $p^{-1}(D)$, since the latter is not a solid torus. If $B = cl(P^2 - D)$, then $p^{-1}(B)$ is the orientable $S^1$-bundle over the Moebius band, and $p^{-1}(\partial B)$ is incompressible in $p^{-1}(B)$. It follows that $p^{-1}(\partial D) = p^{-1}(\partial B)$ is an incompressible torus in M. If $f \approx P^2$ and M has at most one exceptional fiber, then M is homeomorphic to a Seifert fiber space with orbit surface $S^2$ and with three exceptional fibers [12, p. 114].

Therefore assume $f \approx S^2$. If M has at least four exceptional fibers, let $Q_1, \cdots, Q_r$ be simple closed curves, each encircling one of the r exceptional points $P_1, \cdots, P_r$ on f, such that $Q_i \cap Q_j$ is the base point of $\pi_1(f')$, where $f' = cl\left(f - \bigcup U(P_i)\right)$, and such that the curves $Q_1, \cdots, Q_{r-1}$ generate $\pi_1(f')$. Let $\ell$ be a simple closed curve on f' such that $\ell \simeq Q_1 Q_2$ on f'. Then $\pi_1(M)$ has a presentation [10]

$$\pi_1(M) = \{Q_1, \cdots, Q_r, H: [Q_i, H] = 1,\ Q_i^{\alpha_i} H^{\beta_i} = 1,\ Q_1 \cdots Q_r = H^b\},$$

and $\pi_1(M)/\langle H \rangle$ is a free product with amalgamation (over $Q_1 Q_2 = (Q_3 \cdots Q_r)^{-1}$). Here $\langle H \rangle$ denotes the smallest normal subgroup of $\pi_1(M)$ containing H. Therefore $Q_1 Q_2$ has infinite order in $\pi_1(M)/\langle H \rangle$, and since H is an element of the center of $\pi_1(M)$, it follows that $Q_1 Q_2$ and H generate a free abelian subgroup of rank 2 in $\pi_1(M)$. This subgroup is carried by the torus $T = p^{-1}(\ell)$ in M. Hence M is sufficiently large.

If $f = S^2$ and M has at most two exceptional fibers, then M is either a lens space or $S^1 \times S^2$ (cases (1) and (2), respectively).

If $f = S^2$ and M has three exceptional fibers, then M is not sufficiently large if and only if $H_1(M)$ is finite [13, p. 511]. The determinant of the relation matrix for $H_1(M)$ is given by

$$\Delta = b\alpha_1 \alpha_2 \alpha_3 + \beta_1 \alpha_2 \alpha_3 + \beta_2 \alpha_1 \alpha_3 + \beta_3 \alpha_1 \alpha_2$$

(see [10, p. 208]). $H_1(M)$ is finite if and only if the normal form of this matrix has no 0 in the main diagonal, that is, if and only if $\Delta \neq 0$.

## 4. ALMOST SUFFICIENTLY LARGE SEIFERT FIBER SPACES

Suppose M is a Seifert fiber space whose orbit surface f is a closed orientable surface of genus $g \geq 0$, and with n exceptional fibers of orders $\alpha_1, \cdots, \alpha_n$ ($\alpha_i \geq 2$). Then $\pi_1(M)$ has generators

$$A_1, B_1, \cdots, A_g, B_g, Q_1, \cdots, Q_n, H$$

and relations

$$Q_i H Q_i^{-1} = H,$$

$$A_i H A_i^{-1} = H^{\varepsilon_i}, \quad B_i H B_i^{-1} = H^{\eta_i}, \qquad \text{where } \varepsilon_i, \eta_i = \pm 1,$$

$Q_i^{\alpha_i} H^{\beta_i} = 1$ for some relatively prime exponents $\alpha_i$ and $\beta_i$, and

$$Q_1 \cdots Q_n \prod_{i=1}^{g} [A_i, B_i] = H^b, \qquad \text{where b denotes an integer}.$$

It follows that $\pi_1(M)/\langle H \rangle$ has generators

$$A_1, B_1, \cdots, A_g, B_g, Q_1, \cdots, Q_n$$

and relations

$$Q_i^{\alpha_i} = 1,$$

$$Q_1 \cdots Q_n \prod_{i=1}^{g} [A_i, B_i] = 1.$$

The group $\pi_1(M)/\langle H \rangle$ is infinite if and only if either

(i) $g > 0$,

(ii) $n > 3$, or

(iii) $g = 0$, $n = 3$, and $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} \leq 1$ (see [10, p. 203]). In either case, it is a Fuchsian group. If $\pi_1(M)/\langle H \rangle$ has no elements of finite order, then $n = 0$, and M has no exceptional fibers.

THEOREM 1. *A closed,* $P^2$*-irreducible Seifert fiber space* M *with infinite fundamental group has a covering* $\tilde{M}$ *that is an* $S^1$*-bundle over a (closed) orientable surface* F *of genus* $g \geq 1$.

*Proof.* If the orbit surface f of M is nonorientable, we can construct a 2-fold covering of M associated with the orientable cover of f (see [10, p. 198]). Therefore assume that f is orientable. The fiber H of M generates a cyclic normal subgroup $\langle H \rangle$ of $\pi_1(M)$, and $G = \pi_1(M)/\langle H \rangle$ is a Fuchsian group. It is well known (see for example [16, p. 85]) that such a group has a normal subgroup N of finite index that contains no elements of finite order. Let $C: \tilde{M} \to M$ be the covering associated with $\ker \phi$, where $\phi: \pi_1(M) \to G/N$ is the composition $\pi_1(M) \to G \to G/N$. Since $H \in \ker \phi = \pi_1(\tilde{M})$, the fiber lifts to the fiber $\tilde{H}$ of $\tilde{M}$. But the group

$$\pi_1(\tilde{M})/\langle \tilde{H} \rangle \approx \ker \phi/\langle H \rangle \approx N$$

has no elements of finite order; hence $\tilde{M}$ has no exceptional fibers and is an $S^1$-bundle over the orbit surface F. The manifold $\tilde{M}$ is $P^2$-irreducible, and F is a branched covering of f. It follows that F is a closed, orientable surface of genus $g \geq 1$.

COROLLARY. *A* $P^2$*-irreducible Seifert fiber space with infinite fundamental group is almost sufficiently large.*

*Proof.* An $S^1$-bundle over a surface of genus at least 1 is sufficiently large, by the proof of Proposition 2.

*Remark.* After this paper was submitted, I learned that J. Hempel also has a proof of Theorem 1.

## 5. INCOMPRESSIBLE SURFACES IN SEIFERT FIBER SPACES

If the orbit surface f of a Seifert fiber space M is not $S^2$, $P^2$, or a disk, then $p^{-1}(\ell)$ is an incompressible torus or Klein bottle, where $\ell$ is any 2-sided, noncontractible, simple closed curve on f. The following theorem describes the structure of M, in the case where M contains other closed incompressible surfaces.

THEOREM 2. *Suppose* M *is a Seifert fiber space that contains an incompressible surface* F *different from* $S^2$, $P^2$, *torus, Klein bottle, disk, annulus, and Moebius band.*

(a) *If* F *separates* M, *then the two components are twisted* I*-bundles over a compact surface* G, *where* F *is the corresponding* 0*-sphere bundle.*

(b) *If* F *does not separate* M, *then* M *is a fiber bundle over* $S^1$ *with fiber* F.

*Proof.* By Proposition 1, we can assume that M is $P^2$-irreducible. First assume that M is orientable. Then $\pi_1(M)$ has an infinite cyclic normal subgroup

$\langle H \rangle$, generated by the fiber H. Hence M has a 2-fold covering $\tilde{M}$ such that $\pi_1(\tilde{M})$ has nontrivial center. Let $\tilde{F}$ be a component of $p^{-1}(F)$, where $p: \tilde{M} \to M$ is the covering map. Then $\pi_1(\tilde{F})$ does not contain the center of $\pi_1(\tilde{M})$. Hence it follows from Waldhausen's proof of Satz 4.1 in [12] that $\tilde{M}$ is a fiber bundle over $S^1$ with fiber $\tilde{F}$. If $\tilde{F}_1$ is another component of $p^{-1}(F)$, then $\tilde{F}_1$ is parallel to $\tilde{F}$ (see [3, lemma on p. 91]). Thus $\tilde{M}' = cl(\tilde{M} - U(p^{-1}(F)))$ (where $U(\cdots)$ is a small product neighborhood) is homeomorphic to $\tilde{F} \times I$ or consists of two components, each homeomorphic to $F \times I$. Now the theorem follows if we apply [5, Theorem 2] and [6, Theorem 3.2] to the covering $\tilde{M}' \to M' = cl(M - U(F))$.

If M is not orientable, then the 2-fold orientable cover is as in the theorem, and the theorem follows again by the same arguments.

*Remarks.* 1. The proof shows that the hypothesis "M is a Seifert fiber space" can be replaced by "$\pi_1(M)$ has a cyclic normal subgroup". Generalizing [13], C. McA. Gordon and Heil have shown that if M is a sufficiently large orientable 3-manifold and $\pi_1(M)$ contains a cyclic normal subgroup, then M is either a Seifert fiber space or a union of two twisted I-bundles over a closed surface G (see also [15]).

2. For the case where M is orientable, the Seifert fibering in case (b) of Theorem 2 is described in [13, p. 515]. The fibering in case (a) can be constructed in the same way (the fibers are composed of finitely many lines of the two line bundles). This shows again that an incompressible surface F of negative Euler characteristic is isotopic to one for which $p | F: F \to f$ is a branched covering of the orbit surface [12, p. 116].

## REFERENCES


1. H. Boehme, *Irreduzible 3-Mannigfaltigkeiten und ihre 2-blättrigen Überlagerungen.* Dissertation, Berlin, 1969.

2. D. B. A. Epstein, *Projective planes in 3-manifolds.* Proc. London Math. Soc. (3) 11 (1961), 469-484.

3. W. Haken, *Some results on surfaces in 3-manifolds.* Studies in Modern Topology, pp. 39-98. Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.), 1968.

4. W. Heil, *On $P^2$-irreducible 3-manifolds.* Bull. Amer. Math. Soc. 75 (1969), 772-775.

5. ———, *On the existence of incompressible surfaces in certain 3-manifolds. II.* Proc. Amer. Math. Soc. 25 (1970), 429-432.

6. J. Hempel and W. Jaco, *Fundamental groups of 3-manifolds which are extensions.* Ann. of Math. (2) 95 (1972), 86-98.

7. W. Jaco, *The structure of 3-manifold groups* (to appear).

8. G. R. Livesay, *Fixed point free involutions on the 3-sphere.* Ann. of Math. (2) 72 (1960), 603-611.

9. ———, *Involutions with two fixed points on the three-sphere.* Ann. of Math. (2) 78 (1963), 582-593.

10. H. Seifert, *Topologie dreidimensionaler gefaserter Räume*. Acta Math. 60 (1933), 147-238.

11. J. R. Stallings, *Grushko's theorem. II. Kneser's conjecture*. Notices Amer. Math. Soc. 6 (1959), 531-532. Abstract 559-165.

12. F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. II.* Invent. Math. 4 (1967), 87-117.

13. ———, *Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten*. Topology 6 (1967), 505-517.

14. ———, *On irreducible 3-manifolds which are sufficiently large*. Ann. of Math. (2) 87 (1968), 56-88.

15. H. Zieschang, *On extensions of fundamental groups of surfaces and related groups*. Bull. Amer. Math. Soc. 77 (1971), 1116-1119.

16. H. Zieschang, E. Vogt, and H. D. Coldewey, *Flächen und ebene diskontinuierliche Gruppen*. Lecture Notes, vol. 122. Springer-Verlag, New York, 1970.

Florida State University
Tallahassee, Florida 32306