# NONFOLDING MAPS AND THE SINGULAR-REGULAR-NEIGHBORHOOD THEOREM

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## 1. INTRODUCTION

Suppose K is a finite topological complex in a 3-manifold M, and f is a map of M into a 3-manifold N such that f(K) is a topological complex and $K = f^{-1} f(K)$. It is known that f(K) may be tame even though K is not tame [6]. If K is tame and f is a homeomorphism on $M^3 - K$, then f(K) is tame [13]. We show that if K is tame and f is a homeomorphism on K and f "doesn't fold" at any point of f(K), then f(K) is tame (Theorem 4). This is an extension of the singular-regular-neighborhood theorem that was established for surfaces by J. Hempel [12, Theorem 2] and for finite graphs by J. W. Cannon [8]. S. Armentrout showed in [2] that if K is tame and if f is onto N and defines a cellular upper-semicontinuous decomposition of M none of whose nondegenerate elements meets K, then f(K) is tame. Corollary 2 implies that "cellular" may be replaced by "monotone." Theorem 5 is more general than Theorem 4 and Corollary 2; it deals with nonfolding maps that are not necessarily homeomorphisms on either K or $M^3 - K$.

## 2. NOTATION AND TERMINOLOGY

The terms n-*manifold, triangulation, polyhedron,* and *tamely embedded* are used as in [5]. A subset A of an n-manifold M is *cellular* in M if and only if there exists a sequence $C_1, C_2, \cdots$ of n-cells in M such that

(1) for each positive integer i, $C_{i+1} \subset \text{Int } C_i$, and

(2) $\bigcap_{i=1}^{\infty} C_i = A$.

Suppose f is a map of an n-manifold M into an n-manifold N, $p \in N$, and $f^{-1}(p)$ is cellular in M. Choose n-cells B and C in M and N, respectively, such that

$$p \in \text{Int } C, \qquad f^{-1}(p) \subset \text{Int } B, \qquad f(B) \subset \text{Int } C.$$

We say f *folds* (*doesn't fold*) *at* p if and only if $f \mid \text{Bd } B$ is homotopically trivial (nontrivial) in $C - p$. Note that this definition is independent of the choice of n-cells B and C.

Suppose A is a subset of an n-manifold N and f is a map of an n-manifold M into N such that $A \subset f(M)$. We let $G_A$ denote the decomposition of M whose elements are the inverse images of points in A and the singletons of $M - f^{-1}(A)$.

A *topological complex* is a space homeomorphic to a locally finite simplicial complex. Let K be a connected topological complex in a 3-manifold N. We say K has a *singular-regular neighborhood* if there exist a triangulated 3-manifold M, a

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complex K' in Int M, and a map $f: M \to N$ such that $f \mid K'$ is a homeomorphism onto K, $f^{-1}(K) = K'$, and K contains at least one point at which f does not fold. In Theorems 2 and 4, we show that this definition is equivalent to Cannon's [8] in case K is a finite graph. We use the notation L(J, K) for the homology linking number (integer coefficients) of disjoint oriented simple closed curves or loops J and K in $E^3$. We refer the reader to [8] for a discussion of this linking. If X is a compact metric space and f and g are mappings from X into a metric space Y, then $D(f, g) = \sup \{d(f(x), g(x)): x \in X\}$.

## 3. NONFOLDING MAPS

THEOREM 1. *Suppose* A *is a connected subset of an* n-*manifold* N, $p \in A$, *and* f *is a map of an* n-*manifold* M *into* N *such that* $A \subset f(M)$ *and* $G_A$ *is a cellular upper-semicontinuous decomposition of* M. *Let* B *and* C *be* n-*cells such that* $f^{-1}(p) \subset \text{Int } B$ *and* $f(B) \subset \text{Int } C$.

(a) *Then* f *doesn't fold at* p *if and only if*

$$f_*: \pi_{n-1}(B - f^{-1}(p)) \to \pi_{n-1}(C - p)$$

*is one-to-one or, equivalently, if and only if*

$$f_\#: H_{n-1}(B - f^{-1}(p)) \to H_{n-1}(C - p)$$

*is one-to-one (integer coefficients).*

(b) *If* f *doesn't fold at* p, *then* f *doesn't fold at any point of* A.

(c) *If* f *doesn't fold at* p, *then* f(M) *is a neighborhood of* f(A) *in* N.

(d) *If* f *doesn't fold at* p *and* $G_A$ *is a collection of singletons, then* $f \mid f^{-1}(A)$ *is a homeomorphism.*

*Proof.* (a) Clearly, $\pi_{n-1}(B - f^{-1}(p)) = \pi_{n-1}(C - p)$, and this is Z, the group of integers under addition. The map f doesn't fold at p if and only if $f_*(1) \neq 0$, where $f_*$ is the induced map on the homotopy groups.

(b) For each x in A, let $B_x$ denote an n-cell containing $f^{-1}(x)$ in its interior, and let $C_x$ denote an n-cell such that $f(B_x) \subset \text{Int } C_x$. Let H denote the set of points in A at which f folds. Then H is open in A. For if $x \in H$, then there exists a map $F: B_x \to C_x - x$ that extends $f \mid \text{Bd } B_x$. There is an open set U of $f^{-1}(A)$ such that $f^{-1}(p) \subset U \subset \text{Int } B$ and U is the union of elements of $G_A$. Thus

$$(\text{Int } C_x - F(B_x)) \cap f(U)$$

is open in A, and it is contained in H. Also, H is closed in A. For, if $x \in \text{Cl } H$, there is a point y in H such that $f^{-1}(y) \subset \text{Int } B_x$. There exists a map $F: B_x \to C_x - y$ that extends $f \mid \text{Bd } B_x$. Since $C_x$ can be triangulated, there is a polyhedral arc J from y to x that lies in $C_x - f(\text{Bd } B_x)$. Using a sufficiently small regular neighborhood of J in $C_x$, we may push the set $F(B_x)$ off of x by pushing along J without moving $f(\text{Bd } B_x)$. Thus $x \in H$. Since A is connected, H is open and closed in A and $p \notin H$, we conclude that $H = \emptyset$.

(c) Suppose that $x \in A$ and that $B_x$ and $C_x$ are n-cells as in the first line of the preceding paragraph. If $f(B_x)$ is not a neighborhood of x in N, then some polyhedral arc J in $C_x - f(\mathrm{Bd}\, B_x)$ joins x to a point y in $C_x - f(B_x)$. As in the proof of (b), $f(B_x)$ may be pushed off of J. This is impossible, since f does not fold at x, by (b).

(d) Suppose U is open in $f^{-1}(A)$ and $x \in f(U)$. There exists a $B_x$ such that $B_x \cap f^{-1}(A) \subset U$. Thus $f(B_x) \cap A \subset f(U)$. As in the proof of (c), $f(B_x)$ is a neighborhood of x in N. Thus $f \mid f^{-1}(A)$ is open.

THEOREM 2. *Suppose* C *is a 3-cell in* $E^3$, A *is an arc in* C *such that* $A \cap \mathrm{Bd}\, C = \mathrm{Bd}\, A$, *and* A' *is an arc in* Bd C *such that* Bd A' = Bd A. *Suppose* B *is the unit ball in* $E^3$, p *is the origin,* I *is the intersection of* B *and the* y*-axis, and* D *is the intersection of* B *and the* xz*-plane. Suppose also that* f *is a map of* $E^3$ *into* Int C *such that* f *takes* I *homeomorphically into* A *and* $f^{-1}(A) \cap B = I$. *Then the following are equivalent:*

(a) f *does not fold at* f(p),

(b) f | Bd D *is not homologous to zero (integer coefficients) in* Int C - A,

(c) $L(A \cup A', f \mid \mathrm{Bd}\, D) \neq 0$, *and*

(d) *the induced homomorphism* $f_{\#}: H_1(B - I) \to H_1(\mathrm{Int}\, C - A)$ *is not the trivial homomorphism.*

*Proof.* First we show that (b) and (c) are equivalent. Clearly,

$$L(A \cup A', f \mid \mathrm{Bd}\, D) = 0$$

if and only if f | Bd D is null-homologous in $E^3 - (A \cup A')$, and this condition in turn is satisfied if and only if f | Bd D is null-homologous in Int C - A. The latter condition is satisfied because if f | Bd D bounds a singular orientable surface K in $E^3 - (A \cup A')$, we may construct such a surface in Int C - A by cutting off the part of K outside C, filling in the resulting holes in the surface with singular disks in Bd C - (A ∪ A'), and then pushing the resulting surface slightly into Int C - A.

Next we show that (a) implies (b). Let $D_1$ and $D_2$ be the disks in Bd B such that

$$D_1 \cap D_2 = \mathrm{Bd}\, D_1 = \mathrm{Bd}\, D_2 = \mathrm{Bd}\, D.$$

Let $A_1$ and $A_2$ be the subarcs of A such that $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = f(p)$. We may assume that the indices are chosen so that $A_i \cap f(D_i) = \emptyset$ for i = 1 or 2. Let z = f | Bd D. Then z is a singular 1-cycle in C - A. Let $d_i$ denote the singular 2-chain $f \mid D_i$. Then $\partial d_1 = z$ and $\partial d_2 = -z$. Suppose z = 0 in $H_1(C - A)$. Then $z = \partial k$, where k is a 2-chain in C - A. Thus $d_1 - k$ and $d_2 + k$ are 2-cycles in $C - A_2$ and $C - A_1$, respectively. By duality, $C - A_2$ and $C - A_1$ are homologically trivial, so that $d_1 - k = \partial h_1$ and $d_2 + k = \partial h_2$, where $h_1$ is a 3-chain in $C - A_2$ and $h_2$ is a 3-chain in $C - A_1$. Therefore $h_1 + h_2$ is a 3-chain in $C - (A_1 \cap A_2) = C - f(p)$, and $\partial(h_1 + h_2) = d_1 + d_2 = f \mid \mathrm{Bd}\, B$. Thus f folds at f(p).

We show that (b) implies (a). Suppose f folds at f(p). Then f | Bd B is homotopically trivial in C - f(p). Thus there exists a map F: B → C - f(p) that extends f | Bd B. Let $A_1$, $A_2$, $D_1$, and $D_2$ be defined as above. Then $F^{-1}(A_1)$ and $F^{-1}(A_2)$ are disjoint closed sets in B - Bd D, and $F^{-1}(A_i) \cap D_i = \emptyset$, for i = 1 or 2. Hence Bd D bounds an orientable surface in $B - F^{-1}(A_1) - F^{-1}(A_2)$. Hence, f | Bd D is null-homologous in C - A.

Finally, we show (b) and (d) are equivalent. Suppose $f \mid \text{Bd } D$ is homologous to zero. The fundamental 1-cycle on Bd D generates $H_1(B - I) = Z$. Thus $f_\#$ is the trivial homomorphism. It follows immediately that (b) implies (d).

LEMMA 1. *Suppose* B *is a* 3*-cell,* K *is a closed connected set in* Int B, $p \in K$, *and* $f: S^2 \to (\text{Int } B) - K$ *is a map. If* f *is null-homotopic in* B - p, *then* f *is null-homotopic in* B - K.

*Proof.* Suppose f is defined as above, except that f is homotopically nontrivial in B - K. Let U be the component of B - K that contains $f(S^2)$. We consider two cases.

*Case 1.* Suppose Bd B $\not\subset$ U. Since f is homotopically nontrivial in U, J. H. C. Whitehead's sphere theorem [15] implies that there exists a real polyhedral 2-sphere T in U that is nontrivial in U. The sphere T bounds a 3-cell D in Int B. We assert that $K \subset B - D$. For otherwise T = Bd D, and Bd B would be in the same component of B - K, that is, in U. But T is null-homotopic in D, hence in U, and this constitutes a contradiction.

*Case 2.* Suppose Bd B $\subset$ U. In this case, we use the notation of the sphere theorem [15]. Choose a base point q in U. We take the elements of $\pi_2(B - K, q)$ to be equivalence classes of maps from $(S^2, 0)$ into $(B - K, q)$. Let g be a homeomorphism of $(S^2, 0)$ into $(B - K, q)$ such that $K \subset \text{Int } g(S^2)$. Let G be the subgroup of $\pi_2(B - K, q)$ generated by g, and let

$$\Lambda = \{\xi\alpha: \xi \in \pi_1(B - K, q) \text{ and } \alpha \in G\}.$$

Finally, let $\beta = [f] \in \pi_2(B - K, q)$. Now $\beta \notin \Lambda$, because $\beta \neq 0$ and every nonzero element of $\Lambda$ is essential in B - p. Hence, $\Lambda \neq \pi_2(B - K)$. Therefore, there exists a real polyhedral 2-sphere T that is essential in B - K (mod $\Lambda$). But $K \subset \text{Int } T$, hence T is freely homotopic (that is, without restrictions concerning the base point) to $g(S^2)$. Therefore, if $k: S^2 \to B - K$ represents T in $\pi_2(B - K)$, then $[k] \in \Lambda$. This is a contradiction.

THEOREM 3. *Suppose that* A *is a subset of* $E^3$ *and that* $f: M \to E^3$ *is a map of a* 3*-manifold* M *into* $E^3$ *such that* $A \subset f(M)$ *and* $f^{-1}(A)$ *is cellular in* M. *If there exists a point* p *in* A *such that* $f^{-1}(p)$ *is cellular in* M *and* f *doesn't fold at* p, *then* A *is cellular in* $E^3$.

*Proof.* Let C be any 3-cell in $E^3$ containing A in its interior, and let U be any open set in Int C containing A. There is a 3-cell B in M containing $f^{-1}(A)$ in its interior, with $f(B) \subset U$. Since $f \mid \text{Bd } B$ is not null-homotopic in C - p, $f \mid \text{Bd } B$ is not null-homotopic in C - A. Thus, for every x in A, $f \mid \text{Bd } B$ is not null-homotopic in C - x, by Lemma 1. Now $B - f^{-1}(A)$ is connected, since $f^{-1}(A)$ is cellular; thus $f(B - f^{-1}(A))$ is connected. Since A is compact, there exists an open set U' in $E^3$ such that $A \subset U' \subset U$ and C - U' is connected. Let B' be a 3-cell in M such that $f^{-1}(A) \subset \text{Int } B'$ and $f(B') \subset U'$. Then $f \mid \text{Bd } B'$ is not null-homotopic in U' - A. By the sphere theorem [15], some polyhedral 2-sphere S in U' - A is not null-homotopic in U' - A. The sphere S bounds a 3-cell D in C. Since C - U' is connected, $D \subset U'$. Thus $A \subset \text{Int } D \subset D \subset U$.

## 4. TAME EMBEDDINGS

THEOREM 4 (singular-regular-neighborhood theorem). *Suppose* K *is a connected topological complex that is a closed subset of a* 3-*manifold* N *and has a singular-regular neighborhood. Then* K *is tame in* N.

*Proof.* Let M, f, and K' be given by the definition of singular-regular neighborhood. By Theorem 1(b), f doesn't fold at any point of K. Let G be a finite connected graph in the 1-skeleton of K, and let $A_0$ be an arc in G. Suppose that $p \in \text{Int } A_0$ and C is a 3-cell with $p \in \text{Int } C$ and $\text{Bd } A_0 \subset N - C$. Let A denote the subarc of $A_0$ that contains p in its interior and such that $A \cap \text{Bd } C = \text{Bd } A$. Let A' be an arc in Bd C such that Bd A' = Bd A. Let B denote a 3-cell in M containing $f^{-1}(p)$ in its interior such that $f(B) \subset \text{Int } C$, let I denote the subarc of $f^{-1}(A)$ such that $f^{-1}(p)$ is in the interior of I and $I \cap \text{Bd } B = \text{Bd } I$, and let D denote a disk in B that contains p in its interior and is transverse to I. Since f does not fold at p, Theorem 2 implies that $L(A \cup A', f \mid \text{Bd } D) \neq 0$. Thus G has a singular-regular neighborhood as defined in [8], and it is tame [8, Theorem 3.10].

Suppose p is a point in the interior of some 2-simplex $\Delta$ of K. By the local-separation theorem [1, Section 2, Corollary 2], there is a neighborhood U of p in N such that $U - \Delta$ has two components $U_1$ and $U_2$. Let D denote the unit disk in $E^2$, and let h be a homeomorphism of $D \times [-1, 1]$ into M such that

$$f^{-1}(p) \in \text{Int } h(D \times \{0\}) \subset f^{-1}(\Delta), \qquad h[(D \times [-1, 1]) - (D \times \{0\})] \subset M^3 - f^{-1}(\Delta),$$

and $fh(D \times [-1, 1]) \subset U$. Each component of $fh[(D \times [-1, 1]) - (D \times \{0\})]$ lies in either $U_1$ or $U_2$. Suppose both lie in $U_1$. Then $\text{Int } fh(D \times \{0\})$ is locally tame from the $U_1$-side [7, Theorem 6.7.3]. There exists a map g of $fh(D \times [-1, 1])$ into N that pushes $\text{Int } fh(D \times \{0\})$ into $U_1$ and reduces to the identity on $fh(\text{Bd}(D \times [-1, 1]))$, and such that $p \notin gfh(D \times [-1, 1])$. This contradicts the fact that f does not fold at p. Hence, one of the components lies in $U_1$ and the other in $U_2$. As above, it follows that $\Delta$ is locally tame from both sides at f(p). Hence $f(\Delta)$ is locally tame at p. Since a 2-simplex is tame if its interior and boundary are locally tame [14], each 2-simplex in K is tame. Thus K is locally tame [9] and hence tame [5].

COROLLARY 1. *Suppose* K *is a finite connected subcomplex of* $E^3$ *and* $f: E^3 \to E^3$ *is a map that is a homeomorphism on* K *and* $f^{-1}f(K) = K$. *Suppose also that there exists a* 3-*cell* B *containing* K *in its interior such that* $f \mid \text{Bd } B$ *is not homotopic to zero in* $E^3 - f(K)$. *Then* f(K) *is tame.*

*Proof.* Suppose $p \in f(K)$. By Lemma 1, f does not fold at p.

COROLLARY 2. *Suppose* K *is a subcomplex of a* 3-*manifold* M *and* G *is a monotone upper-semicontinuous decomposition of* M *into compact sets such that* M/G *is a* 3-*manifold and* K *misses the nondegenerate elements of* G. *If* P *denotes the projection map of* M *onto* M/G, *then* P(K) *is tame in* M/G.

*Proof.* Suppose that T is a 1-simplex in K and $p \in \text{Int } T$. Let C be a 3-cell in M/G containing P(p) in its interior and small enough so that P(p) lies in some arc A in $C \cap P(T)$ whose endpoints lie in Bd C. There exists a polyhedral 3-cell B in M such that $p \in \text{Int } B \subset B \subset p^{-1}(\text{Int } C)$ and $B \cap T$ is a subarc of T whose endpoints lie in Bd B. Thus f takes $B \cap T$ homeomorphically into A. There exists an open set U in M such that $p \in U \subset \text{Int } B$ and U is the union of elements of the decomposition. Since P(U) is an open neighborhood of P(p) in M/G, there exists a map $g: S^1 \to P(U) - A$ that is not homologous to zero in $\text{Int } C - A$ [1, Section 2, Corollary 2]. By the simplicial-approximation theorem [10], there exists an $\varepsilon > 0$ such that if f: $S^1 \to C$ is a map and $D(f, g) < \varepsilon$, then f and g are homotopic in Int C - A. There exists a map $g_1$: $S^1 \to U - T$ such that $D(Pg_1, g) < \varepsilon$ [3, Lemma 3.2]. Thus $Pg_1$ is not homologous to zero in Int C - A. Hence P does not fold at p, by Theorem 2(d). Thus P(M) is a singular-regular neighborhood of P(K) in M/G.

THEOREM 5. *Let* K *be a connected topological complex that is a closed subset of a* 3*-manifold* N, *let* f: $M \to N$ *be a map of a* 3*-manifold* M *into* N *such that* $K \subset f(M)$, *let* $f^{-1}(K)$ *be a subcomplex of* M, *and let* $G_K$ *be an upper-semicontinuous decomposition of* M *such that* $M/G_K$ *is a* 3*-manifold.*

(a) *If* P *denotes the projection map of* M *onto* $M/G_K$ *and* $fP^{-1}$: $M/G_K \to M$ *doesn't fold at some point of* K, *then* K *is tame.*

(b) *If* K *contains a* 3*-cell, then* K *is tame.*

(c) *If each element of* $G_K$ *is cellular in* M *and* f *doesn't fold at some point of* K, *then* K *is tame.*

(d) *If* f *maps* M *onto* N *and is the projection map of a monotone upper-semicontinuous decomposition of* M *into compact sets, then* K *is tame.*

*Proof.* (a) Let V be a regular neighborhood of $f^{-1}(K)$. Then P(V) is a mapping cylinder neighborhood of P(K) in $M/G_K$, and P(K) is tame in $M/G_K$ [13]. Thus P(K) is a subcomplex of $M/G_K$, under some triangulation of $M/G_K$. Since $fP^{-1}$: $M/G_K \to N$ is a homeomorphism on P(K) and doesn't fold at some point of K, K has a singular-regular neighborhood.

(b) As in part (a), $fP^{-1}$: $M/G_K \to N$ is a homeomorphism on P(K). Thus $fP^{-1}$ cannot fold at a point in the interior of a 3-cell in K. Thus K has a singular-regular neighborhood.

(c) Suppose $p \in K$ and C is a 3-cell containing p in its interior. Let B be a 3-cell in $M/G_K$ containing $Pf^{-1}(p)$ in its interior such that $fP^{-1}(B) \subset \text{Int } C$. Let B' be a 3-cell in M such that $f^{-1}(p) \subset \text{Int } B' \subset B' \subset P^{-1}(\text{Int } B)$. Since f doesn't fold at p, the mapping $f \mid \text{Bd } B'$ is not null-homotopic in C - p. Thus $P \mid \text{Bd } B'$ is not null homotopic in $B - Pf^{-1}(p)$. Thus $(fP^{-1})_*$: $\pi_2(B - Pf^{-1}(p)) \to \pi_2(C - p)$ is not the trivial homomorphism. Thus $fP^{-1}$ doesn't fold at p.

(d) The map $fP^{-1}$: $M/G_K \to N$ is closed and satisfies the hypothesis for the projection map in Corollary 2.

COROLLARY 3. *Let* K *be a connected topological complex that is a closed subset of a* 3*-manifold* N, *let* f: $E^3 \to N$ *be a map such that* $K \subset f(E^3)$, *let* $f^{-1}(K)$ *be a subcomplex of* $E^3$ *that is either a* 2*-manifold or a* 1*-dimensional polyhedron, and let* $G_K$ *be a cellular upper-semicontinuous decomposition of* $E^3$. *If* f *doesn't fold at some point of* K, *then* K *is tame.*

*Proof.* The decomposition space $E^3/G_K$ is $E^3$ [4, Corollaries 4.3 and 4.4]. Thus, by Theorem 5(c), K is tame.

## 5. AN EXAMPLE

In this section, we show that the nonfolding hypothesis in the singular-regular-neighborhood theorem is necessary. Specifically, we give an example of a map f of $E^3$ onto itself that is a homeomorphism on a tame arc $A_1$, and such that $f^{-1}f(A_1) = A_1$ and $f(A_1)$ is wild.

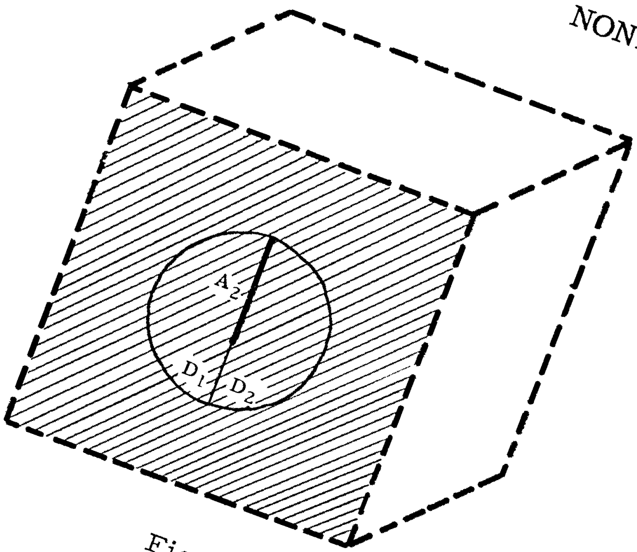


Figure 1

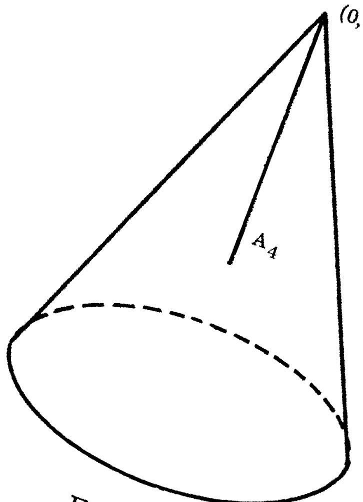


Figure 2

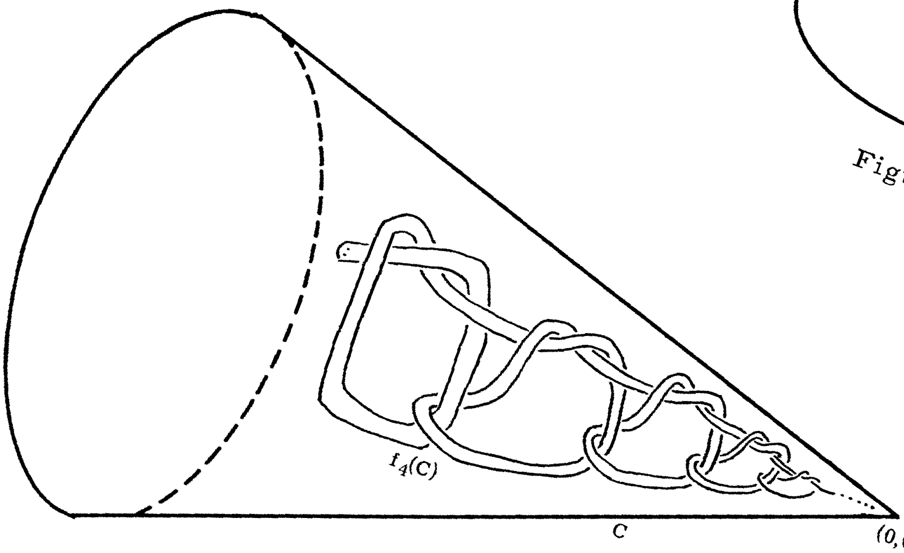


Figure 3

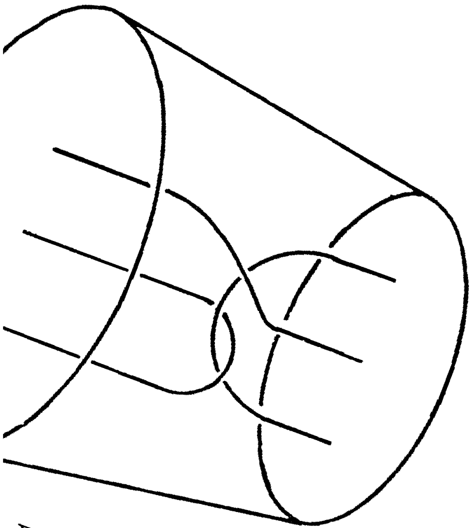

Figure 4

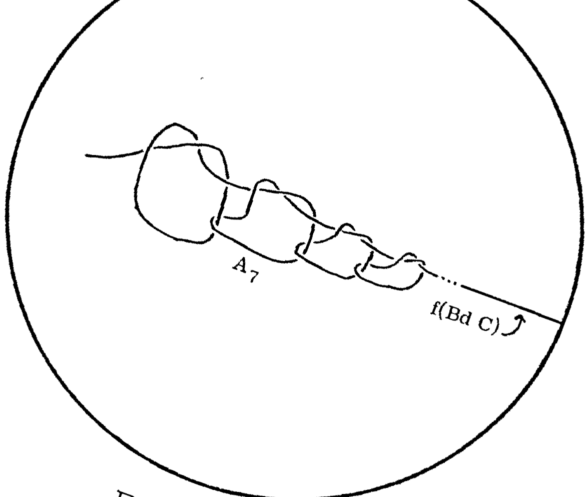


Figure 5

Denote by $E^2$ the plane $z = 0$ in $E^3$, by $E^3_+$ the closed upper half-space of $E^3$, and by $A_1$ a rectilinear interval in $E^2$. We shall describe f as the composite of a finite sequence of maps $f_1$, $f_2$, $f_3$, $f_4$, $f_5$, $f_6$. In what follows, $A_{i+1}$ will denote the arc $f_i(A_i)$. The construction will be made so that $f_i^{-1} f_i(A_i) = A_i$ and $f_i$ is a homeomorphism on $A_i$.

Let $f_1$ be the map $(x, y, z) \to (x, y, |z|)$ of $E^3$ onto $E^3_+$. The map $f_2$ maps $E^3_+$ onto itself so that the intersection of the sets $A_3 = f_2(A_2)$ and $E^2$ is an endpoint of $A_3$. The map $f_2$ is a homeomorphism on $E^3_+ - E^2$ and identifies the disks $D_1$ and $D_2$ in Figure 1 by folding along their intersection. Let C be a cone with open base; that is, let $C = K - D$, where D is the unit disk in $E^2$ and K is the join of D with the point (0, 0, 1). The map $f_3$ is a homeomorphism of $E^3_+$ onto C such that $A_4$ is contained in the centerline of C, and such that the set $A_4 \cap \text{Bd}\, C = (0, 0, 1)$ is an endpoint of $A_4$. See Figure 2. The map $f_4$ is an embedding of C into itself such that $[f_4(C) - (0, 0, 1)] \subset \text{Int}\, C$, $f_4(0, 0, 1) = (0, 0, 1)$, and $A_5$ is a Fox-Artin arc [11, Example 1.2] that is wild at exactly one point, namely (0, 0, 1). See Figure 3.

There exists a map $f_5$ of $f_4(C)$ onto C that is the identity on $A_5$. To show the existence of $f_5$, we divide the cone C into countably many sections, using planes parallel to the base of C. We choose the planes so that each section looks like the one in Figure 4. Let C' be one such section. The map $f_5$ is defined on $C' \cap f_4(C)$ so that $C' \cap f_4(C)$ expands to fill up C'. Note that this may be done so that $f_5^{-1}(f_5(A_5)) = A_5$.

Finally, $f_6$ maps C onto an open 3-cell, it is a homeomorphism on Int C, and it collapses the circles $\text{Bd}\, C \cap \{(x, y, z): z = r\}$ $(0 < r < 1)$ to points. See Figure 5.

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