ON FOURIER TRANSFORMS OF FUNCTIONS IN $H^p(R_+^{n+1})$ FOR $p \leq 1$

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0. INTRODUCTION

For $p > 0$, let $H^p(D)$ denote the space of holomorphic functions f in the unit disk D satisfying the condition

$$\sup_{0 \leq r < 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} = \|f\| \, [H^p] < \infty .$$

In a recent paper [8], C. N. Kellogg proved the following extension of the Hausdorff-Young inequality. Suppose $1 \leq p \leq 2$, and let $1/p + 1/p' = 1$. Then there exists a constant A_p such that for each $f \in H^p(D)$ (with $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$)

$$\left[|\hat{f}(0)|^2 + \sum_{k=0}^{\infty} \left(\sum_{n=2^k}^{2^{k+1}-1} |\hat{f}(n)|^{p'} \right)^{2/p'} \right]^{1/2} \leq A_p \|f\| \, [H^p] . \tag{0.1}$$

The present article originated with an attempt to extend this result to the space $H^p(R_+^{n+1})$ of systems of conjugate harmonic functions in the sense of E. M. Stein and G. Weiss.

Recall that a system $F = (F_0, F_1, \cdots, F_n)$ of $n+1$ harmonic functions in the half-space $R_+^{n+1} = \{(x, y): x \in R^n, \; y > 0\}$ belongs to $H^p(R_+^{n+1})$ for $p \geq (n-1)/n$ (for $p > 0$ if $n = 1$) provided F is the gradient of a harmonic function and

$$\|F\| \, [H^p] = \sup_{y > 0} \left(\int |F(x, y)|^p \, dx \right)^{1/p} < \infty ,$$

where $|F|^2 = \sum_{j=0}^{n} |F_j|^2$. The Fourier transform $\hat{f}$ of a function f in $L^1(R^n)$ is defined by the formula

$$\hat{f}(x) = \int e^{-ix \cdot y} f(y) \, dy ,$$

so that the inverse Fourier transform $f^{\vee}$ is defined by

$$f^{\vee}(x) = (2\pi)^{-n} \int e^{ix \cdot y} f(y) \, dy .$$

The Poisson kernel P for R_+^{n+1} is defined by the equations

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$$P(x, y) = c_n y(y^2 + |x|^2)^{-(n+1)/2}, \qquad c_n = \pi^{-(n+1)/2}\Gamma((n+1)/2),$$

so that $P(\cdot, y)^\wedge(x) = e^{-y|x|}$ (see [12, Chapters 1 and 6]). If $p \leq 1$ and $F \in H^p$, then $F(\cdot, y) \in L^1$ for each $y > 0$; hence the transform $\hat{F}(\cdot, 0)$ is well-defined by the formula $\hat{F}(\cdot, 0) = F(\cdot, y)^\wedge e^{y|\cdot|}$ (see [11]).

Let χ_k denote the characteristic function of the set $\{x: 2^k \leq |x| < 2^{k+1}\}$. The space with mixed norm $L^{(p,q)}(R^n)$ is defined to consist of all measurable functions f on R^n such that

$$\|f\|_{(p,q)} = \|\{\|\chi_k f\|_p\}_{k=-\infty}^{\infty}\|_q < \infty,$$

where the outer norm $\|\ \|_q$ is that of $\ell^q(Z)$ with respect to the counting measure on the set of integers Z. For $p = 1$, Kellogg's result has the following generalization.

PROPOSITION 1. *If* $F \in H^1(R^{n+1}_+)$, *then*

$$\|\hat{F}(\cdot, 0)\|_{(\infty,2)} \leq A\|F\|[H^1]. \tag{0.2}$$

A well-known theorem of G. H. Hardy and J. E. Littlewood asserts that if $p \leq 1$ and a holomorphic function f in the unit disk belongs to the class H^p, then

$$\left(\sum_{n=0}^{\infty} (n+1)^{p-2}|\hat{f}(n)|^p\right)^{1/p} \leq A_p\|f\|[H^p] \tag{0.3}$$

(see [4], for example). T. M. Flett's generalization of another inequality of Hardy and Littlewood (see [4, Theorem 3]) yields a short proof of the following strengthened version of (0.3) for $H^p(R^{n+1}_+)$.

PROPOSITION 2. *For* $(n-1)/n < p < 1$,

$$\left(\int_0^\infty (\sup_{t\leq|x|\leq 2t} |\hat{F}(x, 0)|)^p t^{-n(1-p)-1}\,dt\right)^{1/p} \leq A_p\|F\|[H^p]. \tag{0.4}$$

Furthermore, for $s < \infty$,

$$\int_0^\infty \left(\int_{t\leq|x|\leq 2t} |\hat{F}(x, 0)|^s\,dx\right)^{1/s} t^{-1}\,dt \leq A_s\|F\|[H^1]. \tag{0.5}$$

Clearly (0.4) and (0.5) imply the inequality

$$\left(\int |\hat{F}(x, 0)|^p |x|^{-n(2-p)}\,dx\right)^{1/p} \leq A_p\|F\|[H^p] \qquad ((n-1)/n < p \leq 1).$$

In view of (0.1), it may not be unexpected that the well-known inequality of R. E. A. C. Paley (see [17, Vol. 2, p. 123, Theorem 5.10])

$$\left(\int [\hat{f}^*(x)]^p |x|^{-n(2-p)}\,dx\right)^{1/p} \leq A_p\|f\|_p \qquad (1 < p \leq 2)$$

(where f^* denotes the radial (nonincreasing) equimeasurable rearrangement of f) has the following similar extension to a mixed-norm inequality.

PROPOSITION 3. *If* $1 < p \leq 2$ *and* $s < p'$, *then*

$$\|\hat{f}^*(|\cdot|^{-n(1/p+1/s-1)})\|_{(s,p)} \leq A_{p,s}\|f\|_p . \tag{0.6}$$

In case $p = 1$, the substitute result for H^1 is contained in Proposition 2.

Recall the definition of the space BMO (modulo additive constants) of functions of bounded mean oscillation. We say that $f \in$ BMO if

$$\sup |Q|^{-1}\int_Q |f(x) - \mathrm{av}_Q f|\,dx = \|f\|\,[\mathrm{BMO}] < \infty,$$

where Q ranges through the set of cubes (with sides parallel to the axes), $|Q|$ denotes the volume of Q, and $\mathrm{av}_Q f$ is the average of f over Q. Because of the identification of the dual space of H^1 with the space BMO established by C. Fefferman [3], Propositions 1 and 2 have the following corollary.

PROPOSITION 4. (a) *There exists a constant* A *such that for* $f \in L^{(1,2)}$, *the Fourier transform* $\hat{f}$ *of* f (*in the sense of distributions*) *satisfies the condition*

$$\|\hat{f}\|\,[\mathrm{BMO}] \leq A\,\|f\|_{(1,2)} . \tag{0.7}$$

(b) *Suppose* f *is locally integrable,* $r > 1$, *and* $|\cdot|^{n/r'} f \in L^{(r,\infty)}$; *then* f *is of bounded mean oscillation and*

$$\|\hat{f}\|\,[\mathrm{BMO}] \leq A\,\|\,|\cdot|^{n/r'} f\|_{(r,\infty)} . \tag{0.8}$$

Because the definition of functions of bounded mean oscillation does not involve H^p-spaces, it seems of some interest to give a direct proof of Proposition 4 that does not use Fefferman's result.

1. PROOF OF PROPOSITIONS 1 TO 4

Proposition 1 is a simple consequence of the following result.

LEMMA 1. *Suppose* $\phi \in C^\infty(R^n)$, $0 \leq \phi \leq 1$, *and* $\phi(x) = 1$ *if* $1 \leq |x| \leq 2$, $\phi(x) = 0$ *if* $|x| \leq 1/2$ *or* $|x| \geq 4$; *let* $\phi_k(x) = \phi(2^{-k}x)$ *for* $k \in Z$. *Then*

$$\left\|\left(\sum_{k=-\infty}^{\infty} \{(\hat{F}(\cdot, 0)\,\phi_k)^\vee\}^2\right)^{1/2}\right\|_p \leq A_p\,\|F\|\,[H^p]. \tag{1.1}$$

Proof. Let $m\colon R^n \to \ell^2$ be defined by $m(x) = \{\phi_k(x)\}_{k=-\infty}^{\infty}$; then $D^\alpha \phi_k(x) = 0$ unless $1/2 \leq 2^{-k}|x| \leq 4$, that is, $2^{k-1} \leq |x| \leq 2^{k+2}$; and if $2^j \leq |x| \leq 2^{j+1}$, then

$$|x|^\alpha\,\|m^{(\alpha)}(x)\|_2 \leq 2^{(j+1)|\alpha|}\left(\sum_{|k-j|\leq 2} 2^{-2k|\alpha|}\,\phi^{(\alpha)}(2^{-k}x)^2\right)^{1/2} \leq A_\alpha\,\|\phi^{(\alpha)}\|_\infty .$$

Hence, by a theorem of Stein [11] that clearly extends to multipliers with values in a Hilbert space, m is a multipler from $\mathscr{F}H^p(R_+^{n+1}, C)$ to $\mathscr{F}H^p(R_+^{n+1}, 1^2)$ of norm at most

$$A_p \sup \{ \|\phi^{(\alpha)}\|_\infty : |\alpha| = [\max(n/p, n/2)] + 1 \}.$$

Since

$$(\{\hat{F}(\cdot, 0)\phi_k\}_{k=-\infty}^{\infty})^\vee = \{(\hat{F}(\cdot, 0)\phi_k)^\vee\}_{k=-\infty}^{\infty},$$

this concludes the proof of (1.1).

Proof of Proposition 1. By the Hausdorff-Young inequality, Minkowski's inequality for integrals, and (1.1), it follows that for $1 \le p \le 2$,

$$\begin{aligned}\|\hat{F}(\cdot, 0)\|_{(p',2)} &= \|\{\|\chi_k \hat{F}(\cdot, 0)\|_{p'}\}_{k=-\infty}^{\infty}\|_2 \le \|\{\|\phi_k \hat{F}(\cdot, 0)\|_{p'}\}_{k=-\infty}^{\infty}\|_2 \\ &\le A \|\{\|(\phi_k \hat{F}(\cdot, 0))^\vee\|_p\}_{k=-\infty}^{\infty}\|_2 \\ &\le A \left\| \left(\sum_k \{(\hat{F}(\cdot, 0)\phi_k)^\vee\}^2 \right)^{1/2} \right\|_p \le A_p \|F\| [H^p],\end{aligned}$$

where as in the Introduction the ℓ^p-norm of a sequence is with respect to counting measure. For $p = 1$, this is inequality (0.2).

The argument above indicates that Kellogg's result (0.1) for $1 < p \le 2$ is a corollary of the well-known result of Paley and Littlewood, namely that

$$\left\| \left(\sum_k [(\psi_k \hat{f})^\vee]^2 \right)^{1/2} \right\|_p \le A_p \|f\|_p \quad (1 < p < \infty),$$

where ψ_k denotes the characteristic function of the set $\{x: 2^k \le \max_i |x_i| < 2^{k+1}\}$ (see [9, Theorem 4]). Note also that for $1 < p \le 2$ the nonperiodic n-dimensional analogue $\|f\|_{(p',2)} \le A_p \|f\|_p$ of (0.1) is equivalent to a result of C. S. Herz [6, Lemma 3.1]. (Take $\hat{\kappa}(\xi, h) = \chi_0(|\xi|/|h|)$.)

Proof of Proposition 2. If $p = 1$, let $2 \le s < \infty$; otherwise let $s = \infty$. Let r be equal to the conjugate index s' of s, and let $\chi_{[t,2t]}$ denote the characteristic function of the set $\{x: t \le |x| \le 2t\}$. Then

$$\|\hat{F}(\cdot, 0)\chi_{[t,2t]}\|_s \le e^2 \|\hat{F}(\cdot, 0)e^{-|\cdot|/t}\|_s \le A e^2 \|F(\cdot, t^{-1})\|_r,$$

hence

$$\left(\int_0^\infty \|\hat{F}(\cdot, t)\chi_{[t,2t]}\|_s^p \, t^{-n(1-p)-1} \, dt \right)^{1/p} \le A \left(\int_0^\infty \|F(\cdot, y)\|_r^p \, y^{n(1-p)-1} \, dy \right)^{1/p}$$

Now, by [4, Theorem 3], the last expression is bounded by $A_{r,p} \|F\| [H^p]$.

Proof of Proposition 3. Let τ be some measure-preserving transformation of R^n. It suffices to show that

$$\|\hat{f} \circ \tau |\cdot|^{-(1/p + 1/s - 1)n}\|_{(s,p)} \le A_{p,s} \|f\|_p \quad (1 < p \le 2), \tag{1.2}$$

where $A_{p,s}$ does not depend on τ. Observe that (1.2) is equivalent to the inequality

$$\|\{2^{kn} \|\hat{f} \circ \tau \circ \sigma_k\|_s\}_{k=-\infty}^{\infty}\|_p(\nu) \le A_{p,s} \|f\|_p, \tag{1.3}$$

where

$$\|\{\alpha_k\}\|_p(\nu) = \Big(\sum_k |\alpha_k|^p \nu(k)\Big)^{1/p}, \qquad \nu(k) = 2^{-kn},$$

and σ_k is the mapping from $B = \{x: 1 \le |x| < 2\}$ to $2^k B$ defined by the equation $\sigma_k(x) = 2^k x$.

By Plancherel's theorem, (1.3) holds for $p = s = 2$. Also,

$$\nu(\{k: 2^{kn}\|\hat{f}\circ\tau\circ\sigma_k\|_\infty > \alpha\}) \le \nu(\{k: A2^{kn}\|f\|_1 > \alpha\})$$

$$\le \nu(\{k: 2^{kn} > \alpha/(A\|f\|_1)\}) \le \sum\{2^{-kn}: 2^{kn} > \alpha/(A\|f\|_1)\} \le A\|f\|_1/\alpha .$$

Thus the linear operator taking f to $\{2^{kn}\hat{f}\circ\tau\circ\sigma_k\}_{k=-\infty}^{\infty}$ is bounded between L^2 and $L^{(2,2)}$ and between L^1 and the mixed (quasi-) norm space $L^{(\infty,1\infty)}$ (where 1∞ indicates that the outer quasi-norm is the $L^{1\infty}$- or weak L^1-quasi-norm). By interpolation (see [1], [7]), it follows that

$$\|\{2^{kn}\|\hat{f}\circ\tau\circ\sigma_k\|_{p'}\}\|_{pp'}(\nu) \le A_p\|f\|_p, \tag{1.4}$$

where $\|\cdot\|_{pp'}$ denotes the Lorentz- or $L^{pp'}$-norm (see [1, Section 13.9]) with respect to the measure ν on Z. If now $1 < p \le 2$ and $2 \le s < p'$, let $p_1 = 1$ and $p_2 = s'$, so that $p < p_2 \le 2$; then (1.4) and the inequality

$$\|\hat{f}\circ\tau\circ\sigma_k\|_s \le A\|\hat{f}\circ\tau\circ\sigma_k\|_\infty$$

imply that

$$\|\{2^{kn}\|\hat{f}\circ\tau\circ\sigma_k\|_s\}\|_{p_j\infty}(\nu) \le A\|f\|_{p_j} \qquad (j = 1, 2).$$

The inequality (1.3)—and with it (0.6)—now follows from the interpolation theorem of J. Marcinkiewicz.

Proof of Proposition 4(a). Suppose Q is a cube with center x_0 and side a, where $2^k \le a < 2^{k+1}$. Let

$$f_1(x) = \begin{cases} f(x) & \text{if } |x| \le 2^{-k}, \\ 0 & \text{otherwise,} \end{cases}$$

and let $f_2 = f - f_1$. Then

$$|Q|^{-1}\int_Q |\hat{f}(x) - av_Q\hat{f}|\,dx \le 2|Q|^{-1}\Big(\int_Q |\hat{f}_1(x) - \hat{f}_1(x_0)|\,dx + \int_Q |\hat{f}_2(x)|\,dx\Big); \tag{1.5}$$

also,

$$\int_Q |\hat{f}_1(x) - \hat{f}_1(x_0)|\,dx = \int_Q\int_{|y|\le 2^{-k}} |e^{-iy\cdot x} - e^{-iy\cdot x_0}|\,|f(y)|\,dy\,dx$$

$$\le A\int_Q\int_{|y|\le 2^{-k}} |x - x_0|\,|y|\,|f(y)|\,dy\,dx \le A a^{n+1}\int_{|y|\le 2^{-k}} |y|\,|f(y)|\,dy$$

$$\leq A a^{n+1} \left(\sum_{m=-\infty}^{-k} 2^{2m} \right)^{1/2} \left[\sum_{m=-\infty}^{-k} \left(\int \chi_{m-1}(y)\, |f(y)|\, dy \right)^2 \right]^{1/2}$$

$$\leq A a^{n+1} 2^{-k} \|f\|_{(1,2)} \leq A a^n \|f\|_{(1,2)} .$$

Furthermore,

$$\int_Q |\hat{f}_2(x)|^2\, dx = \sum_{\ell,m=-k}^{\infty} \int_Q \int e^{-ix\cdot y_1} \chi_\ell(y_1) f(y_1)\, dy_1 \int e^{ix\cdot y_2} \chi_m(y_2) \overline{f(y_2)}\, dy_2\, dx$$

$$= \sum_{\ell,m=-k}^{\infty} \int \int \chi_\ell(y_1) \chi_m(y_2) f(y_1) \overline{f(y_2)} \int_Q e^{ix\cdot(y_2-y_1)} dx\, dy_1\, dy_2$$

$$\leq A \left[2^{nk} \sum_{\substack{|\ell-m|\leq 1 \\ \ell,m\geq -k}} \|\chi_\ell f\|_1 \|\chi_m f\|_1 + 2^{(n-1)k} \sum_{\ell,m=-k}^{\infty} 2^{-\max(m,\ell)} \|\chi_\ell f\|_1 \|\chi_m f\|_1 \right]$$

$$\leq A \left[2^{nk} + 2^{(n-1)k} \left(\sum_{\ell,m=-k}^{\infty} 2^{-2\max(m,\ell)} \right)^{1/2} \right] \sum_{\ell=-k}^{\infty} \|\chi_\ell f\|_1^2 \leq A 2^{nk} \|f\|_{(1,2)}^2 .$$

Substitution of these estimates in (1.5) gives the inequality

$$|Q|^{-1} \int_Q |\hat{f}(x) - av_Q \hat{f}|\, dx \leq A \|f\|_{(1,2)} ;$$

this completes the proof of (0.7).

Proof of Proposition 4(b). Note that (1.5) is valid for Q, a, f_1, f_2 defined as before. Next, observe that

$$\int |\hat{f}_1(x) - \hat{f}_1(x_0)|\, dx \leq A a^{n+1} \int_{|y|\leq 2^{-k}} |y|\, |f(y)|\, dy \leq A a^{n+1} \sum_{m=-\infty}^{-k} 2^m \|\chi_{m-1} f\|_1$$

$$\leq A a^{n+1} \sum_{m=-\infty}^{-k} 2^m \sup_m 2^{nm} \left(2^{-nm} \int \chi_{m-1}(y)\, |f(y)|^r\, dy \right)^{1/r}$$

$$\leq A a^n \|\, |\cdot|^{n/r'} f \|_{(r,\infty)} .$$

Also,

$$|Q|^{-1} \int_Q |\hat{f}_2(x)|\, dx \leq \left(|Q|^{-1} \int |\hat{f}_2(x)|^{r'}\, dx \right)^{1/r'} .$$

By the Hausdorff-Young inequality, the right-hand member is at most equal to

$$A|Q|^{-1/r'}\left(\int_{|y|\geq 2^{-k}} |f(y)|^r\,dy\right)^{1/r} = A|Q|^{-1/r'}\left(\sum_{m=-k}^{\infty}\int \chi_m(y)\,|f(y)|^r\,dy\right)^{1/r}$$

$$\leq A|Q|^{-1/r'}\left(\sum_{m=-k}^{\infty} 2^{-nm(r-1)}\right)^{1/r} \|\,|\cdot|^{n/r'} f\|_{(r,\infty)}$$

$$= A|Q|^{-1/r'}\,2^{nk/r'}\,\|\,|\cdot|^{n/r'} f\|_{(r,\infty)} = A\,\|\,|\cdot|^{n/r'} f\|_{(r,\infty)}.$$

Substitution of the preceding estimates in (1.5) completes the proof of inequality (0.8).

2. ADDITIONAL REMARKS

1. Kellogg proved that as a consequence of his extension of the Hausdorff-Young inequality, functions in $L^{(s\infty)}$ are (H^p, H^q)-multipliers, for $1 \leq p \leq 2 \leq q < \infty$ and $1/s = 1/p - 1/q$. As he indicated in his proof of (0.1), the latter result in turn implies the first (if $q = 2$). Observe that these results also are direct consequences of a result of Hardy and Littlewood (see [5, Theorem 14]). The latter has the following extension to R^n. Note that the case $p > 1$ has already been dealt with in [15, Theorem 2] and [14, Appendix (1)].

LEMMA 2. *Supposing* k *belongs to* $L^1 + L^\infty$, *define the function* K *on* R_+^{n+1} *by* $K(x, y) = P(\cdot, y)*k(x)$, *and for* $F \in H^p(R_+^{n+1})$, *define the linear operator* T *by*

$$TF(x, y) = \int_{R^n} F(x - z, y)\,k(z)\,dz\,.$$

Finally, suppose $1 \leq p \leq 2 \leq q < \infty$, *and for* q_0 *defined by* $1/q = 1/p + 1/q_0 - 1$, *suppose* $\|(\partial/\partial y) K(\cdot, y)\|_{q_0} \leq B/y$. *Then* T *is a bounded linear mapping from* H^p *to* H^q, *and*

$$\|TF\|\,[H^q] \leq A_{p,q}\,B\,\|F\|\,[H^p]. \tag{2.1}$$

Proof. The proof is similar to that of Hardy and Littlewood. For every harmonic function G in R_+^{n+1}, set

$$g_k(G)(x) = \left(\int_0^\infty \left|\left(\frac{\partial}{\partial y}\right)^k G(x, y)\right|^2 y^{2k-1}\,dy\right)^{1/2};$$

then (see [12, p. 86]), if $\lim_{y\to\infty} G(x, y) = 0$,

$$\|G\|\,[H^q] \leq A_{q,k}\,\|g_k(G)\|_q\,. \tag{2.2}$$

Also, for $0 < u < y$,

$$T\,F(x, y) = \int F(x - z, y)\,k(z)\,dz = \int F(x - z, y - u)\,K(z, u)\,dz\,;$$

hence

$$\left(\frac{\partial}{\partial y}\right)^2 T\,F(x, y) = \int \frac{\partial}{\partial y} F(x - z, y/2) \frac{\partial}{\partial y} K(z, y/2)\, dz\,.$$

As a result of [12, p. 89] and the main theorem of [10], we obtain the inequality

$$\|g_1(F)\|_p \leq A_p \|F\| \,[H^p]. \tag{2.3}$$

It follows from Minkowski's inequality for integrals and Young's inequality for convolutions that

$$\|g_2(TF)\|_q \leq \left(\int \left\|\int F_y(x - z, y/2)\, K_y(z, y/2)\, dz\right\|_q^2 y^3\, dy\right)^{1/2}$$

$$\leq A\left(\int \|F_y(\,\cdot\,, y)\|_p^2\, y\, dy\right)^{1/2} \leq A\, \|g_1(F)\|_p\,.$$

Hence (2.1) now follows from (2.2) and (2.3).

COROLLARY. *Suppose*

$$g \in L^{(s,\infty)}(R^n), \qquad F \in H^p, \qquad 1 \leq p \leq 2 \leq q < \infty, \qquad 1/s = 1/p - 1/q;$$

then

$$\|(\hat{F}g)^{\vee}\|\, [H^q] \leq A_{p,q}\, \|g\|_{(s,\infty)}\, \|F\|\, [H^p]\,. \tag{2.4}$$

Proof. By Lemma 2, it suffices to observe that for $k = g^{\vee}$ and $1/q = 1/p + 1/q_0 - 1$, the Hausdorff-Young inequality implies that

$$\left\|\frac{\partial}{\partial y} K(\,\cdot\,, y)\right\|_{q_0}^s \leq A^s \int |x|^s e^{-sy|x|}\, |g(x)|^s\, dx$$

$$\leq A^s \sum_{k=-\infty}^{\infty} 2^{s(k+1)} e^{-s2^k y} \int \chi_k(x)\, |g(x)|^s\, dx$$

$$\leq A^s \|g\|_{(s,\infty)}^s \left(\sum_{k < -(\log y)/(\log 2)} + \sum_{k \geq -(\log y)/(\log 2)}\right) \exp s(k \log 2 - 2^k y)$$

$$\leq A^s \|g\|_{(s,\infty)}^s\, y^{-s} \left(\sum_{m=0}^{\infty} \exp(-sm \log 2) + \sum_{m=0}^{\infty} \exp(-s(2^m - m \log 2))\right)$$

$$\leq A_s \|g\|_{(s,\infty)}^s\, y^{-s}\,.$$

2. P. L. Duren, B. W. Romberg, and A. L. Shields [2] have characterized the dual of $H^p(D)$ for $p < 1$ as the space of Lipschitz functions $\Lambda^{1/p-1}$ on the unit circle. There is an extension of this result to $H^p(R_+^{n+1})$ to the effect that the dual of $H^p(R_+^{n+1})$ is topologically isomorphic to $\Lambda^{n(1/p-1)}$ (see [16]). Since the dual of ℓ^p for $p < 1$ is ℓ^∞, Proposition 2 implies that for $\alpha < n$ and for (locally) integrable f,

$$\|f\|\, [\Lambda^\alpha] \leq A_\alpha \|\, |\cdot|^{n\alpha} f\|_{(1,\infty)} \qquad (\alpha > 0), \tag{2.5}$$

where Λ^α is defined to consist of all residue classes of measurable functions g, modulo polynomials of degree at most $[\alpha]$, such that

$$\|g\| [\Lambda^\alpha] = \sup\{|h|^{-\alpha} \|\Delta^k(h)g\|_\infty : h \in R^n\} < \infty,$$

where k denotes the least integer greater than α.

A standard argument yields the following direct proof of (2.5). For each h in R^n $(h \neq 0)$, choose m so that $2^m \leq |h| < 2^{m+1}$. Then

$$\begin{aligned}
|\Delta^k(h)\hat{f}(x)| &= \left|\int (e^{-ih\cdot y} - 1)^k e^{-ix\cdot y} f(y)\, dy\right| \\
&\leq |h|^k \int_{|y|\leq 2^{-m}} |y|^k |f(y)|\, dy + 2^k \int_{|y|\geq 2^{-m}} |f(y)|\, dy \\
&\leq |h|^k \sum_{\ell=-\infty}^{-m-1} 2^{\ell k} \|\chi_\ell f\|_1 + 2^k \sum_{\ell=-m}^{\infty} \|\chi_\ell f\|_1 \\
&\leq \left(|h|^k \sum_{\ell=-\infty}^{-m-1} 2^{\ell(k-\alpha)} + 2^k \sum_{\ell=-m}^{\infty} 2^{-\ell\alpha}\right) \sup_\ell (2^{\ell\alpha} \|\chi_\ell f\|_1) \\
&\leq A|h|^\alpha \| \,|\cdot|^{n\alpha} f\|_{(1,\infty)} .
\end{aligned}$$

3. By duality, Proposition 3 has the following corollary.

PROPOSITION 3'. *Suppose* $2 \leq p < \infty$ *and* f *is locally integrable; then for* $r > p'$,

$$\|\hat{f}\|_p \leq A_{pr} \| \,|\cdot|^{n(1/p'-1/r)} f^*\|_{(r,p)} ,$$

where f^* *again denotes the (nonincreasing) radial rearrangement of* f.

4. Since $\|F(\cdot, y)\|_1 \leq A\|F\| [H^p] y^{-n(1/p-1)}$ for $(n-1)/n \leq p \leq 1$, it follows that

$$|\hat{F}(x, 0)| \leq A e^{y|x|} y^{-n(1/p-1)} \|F\| [H^p] .$$

For $y = |x|^{-1}$, this implies that $|\hat{F}(x, 0)| \leq A\|F\| [H^p] |x|^{n(1/p-1)}$. Thus, in the limiting case $p = (n-1)/n$ (for $n > 1$),

$$|\hat{F}(x, 0)| \leq A\|F\| [H^p] |x|^{n/(n-1)} .$$

In case $n = 1$, consider the space N^p of holomorphic functions f in the upper half-plane $\Pi_+ = \{x + iy: y > 0\}$, satisfying the condition

$$\sup_{y>0} \int_{-\infty}^{\infty} [\log(1 + |f(x+iy)|^{1/p})]^p dx = B < \infty, \tag{2.6}$$

where $\log(1 + |f|^{1/p})$ instead of $\log^+ |f|$ is used to ensure that $f(\cdot + iy) \in L^1$ for each $y > 0$. For $1 \leq p < \infty$ and $\hat{f} = \hat{f}(\cdot + iy) e^{y|\cdot|}$,

$$|\hat{f}(x)| \leq A_p B \exp A_p(B|x|)^{1/(p+1)} . \tag{2.7}$$

This is a consequence of the following lemma.

LEMMA 3. *Suppose* f *is holomorphic in* Π_+, $0 < p < \infty$, *and*

$$\|f(\cdot + iy)\|_1 \leq y \exp(B/y)^{1/p} ; \tag{2.8}$$

then (2.7) *is satisfied. If on the other hand* g *is measurable on* $(-\infty, \infty)$ *and satisfies the condition* $|g(x)| \leq B \exp(B|x|)^{1/(p+1)}$, *where* $0 < p < \infty$, *and if for* $y > 0$,

$$g^{\vee}(x + iy) = \frac{1}{2\pi} \int e^{ixt - y|t|} g(t)\, dt ,$$

then

$$|g^{\vee}(x + iy)| \leq A_p B y^{-1} \exp A_p(B/y)^{1/p} . \tag{2.9}$$

Proof. Condition (2.7) implies that

$$|\hat{f}(x)| = |\hat{f}(\cdot + iy)(x)|\, e^{y|x|} \leq y \exp[(B/y)^{1/p} - y|x|] .$$

The exponent $[(B/y)^{1/p} - y|x|]$ has a minimum for

$$y = p^{-p/(p+1)} B^{1/(p+1)} |x|^{-p/(p+1)} ;$$

hence

$$|\hat{f}(x)| \leq A_p B^{1/(p+1)} |x|^{-p/(p+1)} \exp A_p(B|x|)^{1/(p+1)} \leq A_p B \exp A_p(B|x|)^{1/(p+1)}$$

for $B|x| \geq 1$; on the other hand, if $B|x| \leq 1$, put $y = B^{-1}$ to show that $|\hat{f}(x)| \leq A_p B$. This concludes the proof of (2.7).

To prove the second part of Lemma 3, note that

$$|g^{\vee}(x + iy)|$$

$$\leq A_p B \exp 2^{1/p}(B/y)^{1/p} \int_{|t| \leq 2^{(p+1)/p} B^{1/p} y^{-(p+1)/p}} e^{-y|t|}\, dt + B \int e^{-y|t|/2}\, dt$$

$$\leq A_p B y^{-1} \exp A_p(B/y)^{1/p} .$$

The following more symmetric result for holomorphic functions in the unit disk can be proved similarly.

LEMMA 3'. *Suppose* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *is holomorphic in* D *and* $\alpha > 0$. *Then* $\log|f(z)| = O((1 - |z|)^{-\alpha})$ *as* $|z| \to 1$ *if and only if* $c_n = O(n^{\alpha/(\alpha+1)})$ *as* $n \to \infty$.

To see that condition (2.6) implies (2.8), note that

$$(1/p) \log^+ |f(z)| \leq \log(1 + |f(z)|^{1/p}) .$$

The mean-value inequality for subharmonic functions implies that

$$\log^+ |f(x + iy)| \leq A_p (B/y)^{1/p}$$

(see, for example, [13, p. 84, Section 5.3]); hence $|f(x+iy)| \leq \exp(A_p(B/y)^{1/p})$. Also, $\phi(t) = t/[\log(1+t^{1/p})]^p$ is an increasing function of t for $t \geq 0$; thus for $0 < y \leq B$,

$$\|f(\cdot + iy)\|_1 \leq \phi(\exp[A_p(B/y)^{1/p}])B \leq A_p y \exp[A_p(B/y)^{1/p}].$$

Inequality (2.7) for $f \in N^p$ now follows from Lemma 3.

Furthermore, if f is holomorphic in D and $\int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p d\theta \leq B$ for $0 \leq r < 1$, then, by an argument similar to that above, Lemma 3' implies that $c_n = O(A_p(Bn)^{1/(p+1)})$. To see that the exponent $1/(p+1)$ cannot be replaced by any smaller exponent, consider the function $f(z) = \exp(1-z)^{-\alpha}$ for $0 < \alpha < 1/p$ and α sufficiently close to $1/p$. It is easily verified that $\|f(re^{i\cdot})\|_1 \geq A \exp A(1-r)^{-\alpha}$; hence, by Lemma 3', $c_n \neq O(n^\beta)$ for each $\beta < \alpha/(\alpha+1)$.

5. The replacement of $\hat{F}(\cdot, 0)$ by $\hat{F}(\cdot, 0)^*$ in (0.4) and (0.5) would make the left-hand sides infinite, unless $F = 0$. Thus Proposition 3 does not extend to H^p, for $p \leq 1$. In the case of the unit disk, this does not seem to be equally obvious, and the following example for the case $p = 1$ may therefore be justified.

Let $\phi_n(z) = \sum_{k=0}^n z^k$. Then, by Parseval's equality,

$$\|\phi_n\| [H^2] = (2\pi)^{-1/2}(n+1)^{1/2};$$

hence $\|\phi_n^2\| [H^1] = (n+1)/2\pi$. Moreover, $\phi_n^2(z) = \sum_{k=0}^{2n} (n+1-|n-k|) z^k$, so that the nonincreasing radial rearrangement $\{c_k^*\}$ of the sequence of Taylor coefficients of ϕ_n^2 is given by the rule

$$c_{nk}^* = \begin{cases} n+1-k & \text{for } |k| = 0, 1, \cdots, n, \\ 0 & \text{for } |k| \geq n+1. \end{cases}$$

Also, the relation

$$\sum_{k=0}^{n} c_{nk}^*/(k+1) \sim n \log n \sim \|\phi_n^2\| [H^1] \log n$$

shows that $\{\hat{f}(n)\}$ in (0.3) cannot be replaced by its radial rearrangement. By standard methods, we can use the functions ϕ_n^2 to construct a function $\sum_{n=0}^\infty c_n z^n$ in $H^1(D)$ such that $\sum_{n=0}^\infty c_n^*(n+1) = \infty$.

6. Proposition 4 has natural analogues for periodic functions in R^n and for functions defined on the set of lattice points Z^n of R^n. It appears sufficient to state the version of Proposition 4(b) valid for functions on Z. Suppose $\{c(n)\}_{n=-\infty}^\infty$ is a sequence of complex numbers such that

$$\sup_k 2^k \Big(2^{-k} \sum_{2^k \leq n < 2^{k+1}} |c(n)|^r\Big)^{1/r} = M_r(c),$$

where $r > 1$. Define $\hat{c}$ as the limit in $L^2([-\pi, \pi])$ of the sequence of partial sums $\sum_{n=-k}^{k} c(n)e^{-in\cdot}$ $(k = 0, 1, \cdots)$. Then $\hat{c}$ is of bounded mean oscillation, and

$$\|\hat{c}\|\,[\mathrm{BMO}] \leq A_r M_r(c)\,. \tag{2.10}$$

In the present case, it is even simpler to see that, in contrast to Proposition 3', relation (2.10) is false with $M_r(c^*)$ in place of $M_r(c)$, even if $r = \infty$. For if $c_n(k) = c(k+n)$ for $k \in Z$, then obviously $(c_n)^* = c^*$ but $\hat{c}_n(x) = e^{inx}\hat{c}(x)$. By the Riemann-Lebesgue lemma, the relation $\lim_{|n|\to\infty} a\,v_I\,\hat{c}_n = 0$ holds for each interval I; hence

$$\lim_{|n|\to\infty} \int_I |\hat{c}_n(x) - a\,v_I\,\hat{c}_n|\,dx = \int_I |\hat{c}(x)|\,dx\,.$$

Thus, if $\|\hat{c}\|\,[\mathrm{BMO}] \leq A M_\infty(c^*)$, then

$$\|\hat{c}\|_\infty = \sup_I |I|^{-1} \int_I |\hat{c}(x)|\,dx \leq M_\infty(c^*)\,.$$

The latter inequality, however, is contradicted, for instance, by the sequence $c = \{(|k|+1)^{-1}\}_{k=-\infty}^{\infty}$.

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