ON THE ASYMPTOTIC BEHAVIOR OF FUNCTIONS
HOLOMORPHIC IN THE UNIT DISC

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An asymptotic value of a function \( f \) meromorphic in \( D = \{ |z| < 1 \} \) is defined as a limit value \( \alpha \) of \( f(z) \) as \( |z| \to 1 \) on an arc \( \gamma \) in \( D \). In terms of the associated Riemann surface \( \mathcal{F} \) over the extended complex \( w \)-plane \( \mathcal{W} \), the concept of asymptotic value has the following geometric interpretation: \( \gamma \) is the inverse image of a noncompact arc \( \Gamma \) on \( \mathcal{F} \) whose projection into \( \mathcal{W} \) ends at the point \( w = \alpha \).

A set constitutes the asymptotic set of some meromorphic function \( f \) (that is, the set of asymptotic values of \( f \)) if and only if it is an analytic subset (possibly empty) of \( \mathcal{W} \) (see [1], [2]).

The characterization of the asymptotic sets of holomorphic functions is more difficult, because many analytic sets in \( \mathcal{W} \) must be excluded (see [4], [5]). The following theorem gives a trivial necessary condition. In the statement of the theorem, \( \partial G \) denotes the boundary of \( G \), and the bar \( - \) indicates closure. We can easily verify the necessity of the condition by defining \( G \) as the image of \( D \) under \( f \) and using properties of the Riemann surface of \( f \) (see [4]).

THEOREM. If \( A \) is the asymptotic set of a function \( f \) holomorphic in \( D \), then \( A \) is an analytic set and there exists a domain \( G \) such that:

1. \( \partial G \subset A^- \subset G^- \),
2. if \( \zeta \in \partial G \) is inaccessible from \( G \), then \( \zeta \notin A \),
3. if \( \zeta \in \partial G - A \), then every arc in \( G \) to \( \zeta \) meets \( A \).

The complexities of the holomorphic case are illustrated by the following example, which shows that the condition in the theorem is not sufficient. At the same time, the example answers a question posed by the author [3]. There exists an analytic subset \( A \) of \( \{ |w| < 1 \} \) that meets every arc in \( \{ |w| < 1 \} \) ending at a point of \( \{ |w| = 1 \} \) but is not the asymptotic set of any function holomorphic in \( D \).

To construct the example, let \( S \) be the finite domain bounded by the triangle with vertices at \((0, 1/4), (0, -1/4), \) and \((1, 0)\). Define

\[
C_n = \{ |w| = 1 - 2^{-n} \} \quad \text{and} \quad C_{n,m} = \{ |w| = 1 - 2^{-n} + 2^{-m} \}.
\]

Now put

\[
A_n = \{C_n - S\} \cup \bigcup_{m \geq n+2} \{C_{n,m} \cap S\}
\]

and

\[
A = \bigcup_{n=1}^{\infty} A_n \cup \{0\}.
\]

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It is clear that $A$ is a Borel set, hence analytic [6]. Moreover, $A$ is a subset of $\{|w| < 1\}$ that meets every arc in $\{|w| < 1\}$ ending at $\{|w| = 1\}$.

The inclusion of $\{0\}$ in the set $A$ ensures that if $A$ is the asymptotic set of a function $f$ holomorphic in $D$, then the Riemann surface $\mathcal{S}$ of $f$ covers $\{|w| < 1\} - A$. This implies the existence of a noncompact arc $\Gamma$ on $\mathcal{S}$ whose projection is an arc in $\{|w| < 1\}$ that ends at $w = 1$. To obtain $\Gamma$, define $w_n = 1 - 2^{-n}$ and let $\gamma_1$ be an arc in $S - A$ from $w = 1/4$ to $w_1$. Since $w_1 \not\in A$, $\gamma_1$ can be lifted completely into $\mathcal{S}$, determining there a compact arc $\Gamma_1$ ending at a point $P_1$ of $\mathcal{S}$ over $w_1$. Now $\mathcal{S}$ contains a neighborhood of $P_1$, and $w_2 \not\in A$; consequently, some arc $\gamma_2$ joining $w_1$ to $w_2$ in $S$ can be lifted completely into $\mathcal{S}$, determining there an arc $\Gamma_2$ beginning at $P_1$ and ending at a point $P_2$ of $\mathcal{S}$ over $w_2$. Since $w_n \not\in A$ for any $n$, we can repeat the construction. The result is a non-compact arc $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots$ on $\mathcal{S}$ whose projection $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots$ is an arc in $S$ ending at $w = 1$; this implies $1 \in A$, a contradiction. Consequently, $A$ cannot be the asymptotic set of a function holomorphic in $D$.

It is interesting to note that a simple alteration in the definition of $A$ produces an analytic set $A'$ that has the same pathological behavior as $A$, but can be realized as the asymptotic set of a function holomorphic in $D$. Specifically, set

\[ C_{n,m}' = \{|w| = 1 - 2^{-n} - 2^{-m}\}, \]

and put

\[ A_n' = \{C_n - S\} \cup \bigcup_{m \geq n+2} \{C_{n,m}' \cap S\} \]

and

\[ A' = \bigcup_{n=1}^{\infty} A_n'. \]

Using as sheets the finite domains bounded by the $A_n'$, we can construct a Riemann surface over the unit disc such that $A'$ is the asymptotic set of the corresponding holomorphic function in $D$.

REFERENCES


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