APPLICATIONS OF EXTREME-POINT THEORY TO UNIVALENT FUNCTIONS

Thomas H. MacGregor

1. INTRODUCTION

We shall show how the knowledge of the extreme points of a family of analytic functions can be an effective means of solving extremal problems. The extremal problems considered are not always linear, and our families consist primarily of various kinds of univalent functions. The most striking of our new results deal with close-to-convex functions. We indicate how our approach is useful in a variety of situations, and how it affords a systematic treatment of several classical results in the theory of univalent functions.

Let $\mathscr{A}$ denote the set of all functions analytic in the unit disk $\Delta = \{z: |z| < 1\}$. Then $\mathscr{A}$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of Δ. Let S denote the subset of $\mathscr{A}$ consisting of the univalent functions that satisfy the normalization conditions $f(0) = 0$ and $f'(0) = 1$. Also, let St, K, C, and R denote the subsets of S consisting of starlike, convex, close-to-convex, and real mappings, respectively, so that, for example, $f \in \mathrm{St}$ or $f \in K$ if the domain $f(\Delta)$ is starlike (with respect to the origin) or convex, and $f \in R$ if $f(z)$ is real when z is real $(-1 < z < 1)$. The family C was introduced in [10] by W. Kaplan, and it can be described in terms of a geometric mapping property. Analytically, a function f is in C if there exist a function g and a complex number a such that $ag \in \mathrm{St}$ and $\Re\{zf'(z)/g(z)\} > 0$ for $|z| < 1$. We recall the inclusion relations $K \subset \mathrm{St} \subset C$.

We shall discuss some observations made in [3] by L. Brickman, D. R. Wilken, and this author. Let $\mathfrak{H}B$ denote the closed convex hull of the set B; also, let $\mathfrak{E}(\mathfrak{H}B)$ denote the extreme points of $\mathfrak{H}B$. Each of the four families St, K, C, and R is locally uniformly bounded, because S has this property. In fact, each family is even compact. This implies that $\mathfrak{H}\mathrm{St}$, $\mathfrak{H}K$, $\mathfrak{H}C$, and $\mathfrak{H}R$ are compact. Consequently (see [5, p. 440])

$$\mathfrak{E}(\mathfrak{H}\mathrm{St}) \subset \mathrm{St}, \quad \mathfrak{E}(\mathfrak{H}K) \subset K, \quad \mathfrak{E}(\mathfrak{H}C) \subset C, \quad \mathfrak{E}(\mathfrak{H}R) \subset R.$$

These four sets of extreme points were completely determined in [3], as follows:

$$\mathfrak{E}(\mathfrak{H}\mathrm{St}) = \{f: f(z) = z/(1 - \varepsilon z)^2 \quad (|\varepsilon| = 1)\},$$

$$\mathfrak{E}(\mathfrak{H}K) = \{f: f(z) = z/(1 - \varepsilon z) \quad (|\varepsilon| = 1)\},$$

$$\mathfrak{E}(\mathfrak{H}C) = \left\{f: f(z) = \left[z - \frac{1}{2}(\varepsilon + \delta)z^2\right]/[1 - \delta z]^2 \quad (|\varepsilon| = |\delta| = 1,\ \varepsilon \neq \delta)\right\},$$

$$\mathfrak{E}(\mathfrak{H}R) = \{f: f(z) = z/(1 + bz + z^2) \quad (-2 \leq b \leq 2)\}.$$

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It was also shown there that $\mathfrak{H}K$ consists of the functions f that are analytic in Δ and satisfy the conditions $\Re\{f(z)/z\} > 1/2$, $f(0) = 0$, and $f'(0) = 1$. Also, $\mathfrak{H}R$ consists of the class T of typically real functions, introduced in [22] by W. Rogosinski; that is,

$$\mathfrak{H}R = T = \{f\colon f \in \mathscr{A},\ f(0) = 0,\ f'(0) = 1, \text{ and}$$
$$f(z) \text{ is real if and only if } z \text{ is real } (-1 < z < 1)\}.$$

If J is a complex-valued, continuous, linear functional on $\mathscr{A}$, then

$$\max_{f \in \mathfrak{H}\mathscr{F}} |J(f)| = \max_{f \in \mathscr{F}} |J(f)| = \max_{f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})} |J(f)|,$$

where $\mathscr{F}$ denotes any one of the families St, K, C, and R. This was pointed out in [3, pp. 99 and 100] in slightly different form; and we shall use similar relations in some of the developments here. The exact determination of the set $\mathfrak{E}(\mathfrak{H}\mathscr{F})$ for each of these families shows that the problem of maximizing $|J(f)|$ over any one of them is reduced to the problem of maximizing $|J|$, where J becomes an analytic function of one or two variables represented by the parameters ε, δ, and b. By this technique, one can deduce various known results about linear extremal problems over St, K, C, or R (see [3, p. 99]); we hope this approach will also lead to the solution of other linear problems.

Our approach gains further significance from its applicability to certain nonlinear extremal problems. We also use our method to solve extremal problems over families of analytic functions related to St, K, C, or R. In particular, we obtain the following two specific results. If $f \in C$ and $0 < r < 1$, then

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \le \frac{r}{1 - r^2}.$$

If $f(z) = \sum_{n=1}^{\infty} a_n z^n$ in Δ, and if $F \in C$ and $|f(z)| \le |F(z)|$ in Δ, then $|a_n| \le n$ for $n = 1, 2, \cdots$.

The first result quoted above was proved by M. S. Robertson in [18] for the smaller class St. It is an open problem whether it holds for all f in S. In this direction, it is known that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta < \frac{r}{1 - r} \quad \text{if } f \in S$$

(see [8, p. 10]), and this has been improved (for $r \ge r_0$) by I. E. Bazilevič in [1] to

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta < \frac{r}{1 - r^2} + 0.55.$$

We also obtain sharp inequalities for each L^p-means of functions in C for $p = 1, 2, 3, \cdots$. In addition, we find the precise upper bounds on the L^p-means of the nth derivative of functions in C for $n = 1, 2, \cdots$ and all $p \ge 1$. For the case where the majorant function F belongs to St or R, the author proved the inequality $|a_n| \le n$ in an earlier paper [14].

2. COEFFICIENT ESTIMATES FOR FUNCTIONS ASSOCIATED WITH UNIVALENT FUNCTIONS THROUGH MAJORIZATION OR THROUGH SUBORDINATION

Suppose that f and g are analytic in Δ and $|f(z)| \leq |g(z)|$ for each z in Δ. We then say that f is *majorized* by g in Δ. In [14], we conjectured that if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is majorized by a function g in S, then $|a_n| \leq n$ for $n = 1, 2, \cdots$. This generalization of the Bieberbach conjecture was proved for the special cases where $g \in St$ (more generally, where g is spiral-like) or $g \in T$. We now prove it for the case where $g \in C$.

Let $\mathscr{G}$ denote a compact subset of S, and let $\mathscr{F}$ denote the subset of $\mathscr{A}$ consisting of the functions f that are majorized in Δ by some g in $\mathscr{G}$. The functions f and g are then related by a relation $f = \phi g$, where ϕ is an analytic function in Δ such that $|\phi(z)| \leq 1$. The inclusion relation $\mathscr{G} \subset S$ implies that $\mathscr{G}$ is locally uniformly bounded; therefore, $\mathscr{F}$ is locally uniformly bounded. Hence, $\mathfrak{H}\mathscr{F}$ is compact and, in particular, this implies that $\mathfrak{E}(\mathfrak{H}\mathscr{F}) \subset \mathscr{F}$.

To see that $\mathscr{F}$ is even compact, suppose that $\{f_n\}$ is a sequence in $\mathscr{F}$, and write $f_n = \phi_n g_n$. After extracting a subsequence, we may assume that $g_n \to g$, where $g \in \mathscr{G}$. Likewise, we may assume that $\phi_n \to \phi$. The limits $g_n \to g$ and $\phi_n \to \phi$ are uniform for $|z| \leq r$ $(0 < r < 1)$. We need only show that $\phi_n g_n \to \phi g = f$ uniformly in $|z| \leq r$, since $f \in \mathscr{F}$. If $|z| \leq r$, then

$$\begin{aligned}|\phi_n(z) g_n(z) - \phi(z) g(z)| &\leq |\phi_n(z) g_n(z) - \phi_n(z) g(z)| + |\phi_n(z) g(z) - \phi(z) g(z)| \\ &\leq |g_n(z) - g(z)| + M(r)\,|\phi_n(z) - \phi(z)|\,,\end{aligned}$$

where $M(r) = \max_{|z| \leq r} |g(z)|$. The compactness of $\mathscr{F}$ is now clear.

Let J be a complex-valued, continuous, linear functional on $\mathscr{A}$, and let

$$\mathscr{F}_0 = \{f\colon f \in \mathfrak{H}\mathscr{F} \text{ and } |J(f)| = \max_{F \in \mathfrak{H}\mathscr{F}} |J(F)|\}.$$

Because the class $\mathscr{F}_0$ is compact and nonvoid, it contains an extremal element [5, p. 439]. The set $\mathscr{F}_0$ is an extremal set, because of the linearity of J; that is, if $0 < t < 1$ and $tf + (1-t)g \in \mathscr{F}_0$, then $f \in \mathscr{F}_0$ and $g \in \mathscr{F}_0$. To see this, let $M = \max_{F \in \mathfrak{H}\mathscr{F}} |J(F)|$, and assume that $tf + (1-t)g \in \mathscr{F}_0$, where $f \in \mathfrak{H}\mathscr{F}$ and $g \in \mathfrak{H}\mathscr{F}$; then

$$\begin{aligned}M = |J(tf + (1-t)g)| &= |tJ(f) + (1-t)J(g)| \\ &\leq t|J(f)| + (1-t)|J(g)| \leq tM + (1-t)M = M.\end{aligned}$$

Thus, $|J(f)| = |J(g)| = M$, and therefore $f \in \mathscr{F}_0$ and $g \in \mathscr{F}_0$. This implies that $f_0 \in \mathfrak{E}(\mathfrak{H}\mathscr{F})$; that is, there exists an extreme point that maximizes $|J(f)|$ over $\mathfrak{H}\mathscr{F}$. Combining this with some of the previous observations, we conclude that

$$\max_{f \in \mathscr{F}} |J(f)| = \max_{f \in \mathfrak{H}\mathscr{F}} |J(f)| = \max_{f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})} |J(f)|.$$

Suppose that $f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})$. Then $f \in \mathscr{F}$, and thus $f = \phi g$, where $g \in \mathscr{G}$ and ϕ is analytic in Δ and satisfies the inequality $|\phi(z)| \leq 1$. We claim that $g \in \mathfrak{E}(\mathfrak{H}\mathscr{G})$.

Otherwise, we may write $g = th + (1-t)k$, where $0 < t < 1$, $h \neq k$, and h and k belong to $\mathfrak{H}\mathscr{G}$. This implies that $f = tf_1 + (1-t)f_2$, where $f_1 = \phi h$ and $f_2 = \phi k$. Since h and k belong to $\mathfrak{H}\mathscr{G}$, it follows that f_1 and f_2 belong to $\mathfrak{H}\mathscr{F}$. The equation $0 = \frac{1}{2}g + \frac{1}{2}(-g)$ shows that the function 0 does not belong to $\mathfrak{E}(\mathfrak{H}\mathscr{F})$, and thus $\phi \neq 0$. By the uniqueness theorem for analytic functions, the conditions $\phi \neq 0$ and $h \neq k$ imply that $f_1 \neq f_2$. Thus, the relation $f = tf_1 + (1-t)f_2$ actually contradicts that $f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})$.

The arguments above show that $\max_{f \in \mathscr{F}} |J(f)| = \max_{f \in \mathscr{F}_0} |J(f)|$, where $\mathscr{F}_0$ consists of the functions f in $\mathscr{F}$ for which there exist analytic functions g and ϕ ($g \in \mathfrak{E}(\mathfrak{H}\mathscr{G})$, $|\phi(z)| \leq 1$ in Δ) such that $f = \phi g$.

LEMMA. *Let* $f(z) = (1-z)^{1/2} = \sum_{n=0}^{\infty} b_n z^n$ ($|z| < 1$), *and let* $f_n(z) = \sum_{k=0}^{n} b_k z^k$. *Then* $|f_n(z)| \leq 1$ *if* $\left|z - \frac{1}{2}\right| \leq \frac{1}{2}$ ($n = 0, 1, 2, \cdots$).

Proof. The coefficients of the power series for f are given by the binomial series, and we simply note that $b_0 = 1$, $b_1 = -1/2$, and $b_n < 0$ for $n = 2, 3, \cdots$.

Let $c_n = -b_n$ for $n \geq 2$, and recall that the series converges at $z = 1$ as well as for $|z| < 1$. By the maximum-modulus theorem, we need only show that $|f_n(z)| \leq 1$ if $\left|z - \frac{1}{2}\right| = \frac{1}{2}$, and thereafter we may set $z = \frac{1}{2} + \frac{1}{2}e^{i\theta} = e^{i\theta/2}\cos\theta/2 = e^{i\alpha}\cos\alpha$, where α is real. Now

$$|f_n(z)| = \left|1 - \frac{1}{2}z - c_2 z^2 - c_3 z^3 - \cdots - c_n z^n\right| \leq \left|1 - \frac{1}{2}z\right| + \sum_{k=2}^{n} c_k |z|^k$$

$$\leq \left|1 - \frac{1}{2}z\right| + \sum_{k=2}^{\infty} c_k |z|^k = \left|1 - \frac{1}{2}z\right| + 1 - \frac{1}{2}|z| - (1 - |z|)^{1/2}$$

$$= \left(1 - \frac{3}{4}\cos^2\alpha\right)^{1/2} + 1 - \frac{1}{2}|\cos\alpha| - (1 - |\cos\alpha|)^{1/2}.$$

It is an easy matter to verify the inequality

$$\left(1 - \frac{3}{4}x^2\right)^{1/2} + 1 - \frac{1}{2}x - (1-x)^{1/2} \leq 1 \quad \text{for } 0 \leq x \leq 1.$$

THEOREM 1. *Suppose that* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *is analytic and that* $|f(z)| \leq |g(z)|$ *for* $|z| < 1$, *where* g *is close-to-convex for* $|z| < 1$, $g(0) = 0$, *and* $g'(0) = 1$. *Then* $|a_n| \leq n$ *for* $n = 1, 2, 3, \cdots$.

Proof. Our introductory remarks show that we need only prove that $|a_n| \leq n$, assuming that $f = \phi g$, where ϕ is analytic, $|\phi(z)| \leq 1$, and $g \in \mathfrak{E}(\mathfrak{H}C)$. According to the result obtained in [3], g must be a function of the form

$$g(z) = \left[z - \frac{1}{2}(\varepsilon + \delta)z^2\right] / [1 - \delta z]^2, \quad \text{where } |\varepsilon| = |\delta| = 1 \text{ and } \varepsilon \neq \delta.$$

In [14], we showed that the relation $f = \phi g$ with $|\phi(z)| \leq 1$ always implies that $|a_n| \leq 1 + |B_3|^2 + |B_5|^2 + \cdots + |B_{2n-1}|^2$, where

$$G(z) = \sqrt{g(z^2)} = \sum_{n=1}^{\infty} B_{2n-1} z^{2n-1} \qquad (B_1 = 1).$$

We shall obtain the conclusion $|a_n| \leq n$ by showing that $|B_{2n-1}| \leq 1$ for $n = 1, 2, 3, \cdots$, and we need do this only for the functions g mentioned above.

Write $\varepsilon = a\delta$, so that $|a| = 1$ and $a \neq 1$, and choose η so that $\eta^2 = \delta$. If the function H is defined by the equation $zH(z^2) = \eta G(z/\eta)$, then

$$H(z) = (1 - bz)^{1/2}/(1 - z),$$

where $b = (1 + a)/2$. We also find that $D_n = b_0 + b_1 b + b_2 b^2 + \cdots + b_n b^n$, if the power series for H is $H(z) = \sum_{n=0}^{\infty} D_n z^n$ and the numbers $\{b_n\}$ are defined as in the lemma. By the lemma, we conclude that $|D_n| \leq 1$, since $D_n = D_n(b)$ may be identified with f_n and b satisfies the condition $|b - 1/2| = 1/2$. Also, $|B_{2n-1}| = |D_{n-1}|$, because $|\eta| = 1$; and this completes the proof.

Extreme-point theory is well adapted to a number of problems illustrated by Theorem 1. For example, in [14] we showed that $|a_n| \leq n$ if f is majorized by some function g in St. This result is contained in Theorem 1; but it can also be viewed as a consequence of the fact that if $g(z) = z/(1 - \varepsilon z)^2$ ($|\varepsilon| = 1$), then $G(z) = \sqrt{g(z^2)} = z/(1 - \varepsilon z^2)$ and thus $|B_{2n-1}| = 1$ for all n. The proof uses the fact that the set of functions g described above is precisely the set $\mathfrak{E}(\mathfrak{H}\,\mathrm{St})$.

If f is majorized by some function g in K, then to determine $\max|a_n|$ we need only consider functions of the form $f = \phi g$, where $\phi(z) = \sum_{n=0}^{\infty} c_n z^n$ is analytic and $|\phi(z)| \leq 1$ for $|z| < 1$, and where g has the form $g(z) = z/(1 - \varepsilon z)$ ($|\varepsilon| = 1$). Multiplying the power series for ϕ and g, we find that

$$a_n = c_0 \varepsilon^{n-1} + c_1 \varepsilon^{n-2} + \cdots + c_{n-2}\varepsilon + c_{n-1} = \varepsilon^{n-1}(c_0 + c_1\gamma + c_2\gamma^2 + \cdots + c_{n-1}\gamma^{n-1}),$$

where $\gamma = 1/\varepsilon$ and thus $|\gamma| = 1$. This clearly shows that the problem of maximizing $|a_n|$ over $\{f\}$ is equivalent to the problem of maximizing the modulus of the partial sums of the power series of an arbitrary bounded analytic function ϕ. Thus, by Landau's bound on the partial sums (see [12, p. 20]), $|a_n| \leq G_n$, a relation that our proof in [14] fails to make clear. We also note that our earlier arguments imply that these results hold not only if $g \in \mathcal{G}$, but also for the larger class where $g \in \mathfrak{H}\mathcal{G}$. For example, it follows that $|a_n| \leq G_n$ if f is majorized by a function g that is analytic in Δ and satisfies the conditions $\Re\{g(z)/z\} > 1/2$, $g(0) = 0$, and $g'(0) = 1$. This more general result was indicated by some of the considerations of Robertson in [20].

Extreme-point theory can also be applied to linear extremal problems associated with subordination. Suppose that $\mathcal{G}$ is a compact subset of S, and that $\mathcal{F}$ is the class of functions subordinate to some function in $\mathcal{G}$. That is, suppose that f is analytic in Δ and $f(z) = g(\phi(z))$, where ϕ is analytic in Δ, $|\phi(z)| < 1$, $\phi(0) = 0$, and $g \in \mathcal{G}$. Since $\mathcal{G}$ is locally uniformly bounded and $|\phi(z)| \leq |z|$ by Schwarz's lemma, it follows that $\mathcal{F}$ is locally uniformly bounded (in fact, $|f(z)| \leq |z|/(1 - |z|)^2$ if $f \in \mathcal{F}$). Thus, $\mathfrak{H}\mathcal{F}$ is compact, and again we find that $\mathfrak{E}(\mathfrak{H}\mathcal{F}) \subset \mathcal{F}$.

We can also show that $\mathcal{F}$ is compact. The argument is similar to the one given for majorization. We need only show that if $g_n \to g$ and $\phi_n \to \phi$ uniformly for $|z| \leq r$, then $|g_n(\phi_n(z)) - g(\phi_n(z))|$ and $|g(\phi_n(z)) - g(\phi(z))|$ can be made arbitrarily small (for $|z| \leq r$) with large n. The first expression is small because the convergence $g_n \to g$ is uniform and $|\phi_n(z)| \leq |z|$. The second expression is small because the convergence $\phi_n \to \phi$ is uniform, ϕ_n and ϕ satisfy the conditions of Schwarz's lemma, and the family $\mathscr{G}$ is equicontinuous.

Suppose that $f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})$. Then $f \in \mathscr{F}$, and as above we may write $f(z) = g(\phi(z))$. We claim that either $f = 0$ or $g \in \mathfrak{E}(\mathfrak{H}\mathscr{G})$. Suppose that, to the contrary, $f \neq 0$ and $g \notin \mathfrak{E}(\mathfrak{H}\mathscr{G})$. Then $g = th + (1 - t)k$, where $0 < t < 1$, $h \neq k$, and h and k belong to $\mathfrak{H}\mathscr{G}$. This implies that

$$f = tf_1 + (1 - t)f_2, \quad \text{where } f_1(z) = h(\phi(z)) \text{ and } f_2(z) = k(\phi(z)).$$

Since $h \in \mathfrak{H}\mathscr{G}$, it follows that $f_1 \in \mathfrak{H}\mathscr{F}$, because of Schwarz's lemma and the meanings of $\mathfrak{H}\mathscr{G}$ and $\mathfrak{H}\mathscr{F}$. Likewise, $f_2 \in \mathfrak{H}\mathscr{F}$. Because $f \neq 0$, we see that $\phi \neq 0$, and thus ϕ is an open map. Therefore, $f_1 \neq f_2$, since otherwise the relation $h(w) = k(w)$ would hold on some open set, and this would imply that $h = k$.

Let J be any complex-valued, continuous, linear functional on $\mathscr{A}$. Since $J(0) = 0$, the previous remarks imply that

$$\max_{f \in \mathscr{F}} |J(f)| = \max_{f \in \mathfrak{H}\mathscr{F}} |J(f)| = \max_{f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})} |J(f)|$$

$$= \max \{ |J(f)| : f = g(\phi), \text{ where } g \in \mathfrak{E}(\mathfrak{H}\mathscr{G}), \ \phi \text{ is analytic in } \Delta,$$

$$|\phi(z)| < 1, \text{ and } \phi(0) = 0 \}.$$

We shall illustrate this technique by showing that the inequality $|a_n| \leq n$ holds for the coefficients of a function f subordinate to some function g in St (this is not new; it was proved by W. Rogosinski in [23]). We may assume that g has the form $g(z) = z/(1 - \varepsilon z)^2$, where $|\varepsilon| = 1$, since these are the functions in $\mathfrak{E}(\mathfrak{H}\,\mathrm{St})$. Then

$$f(z) = \frac{1}{4}(p^2(\phi(z)) - 1), \quad \text{where } p(z) = \frac{1 + \varepsilon z}{1 - \varepsilon z}.$$

Since the function $q(z) = p(\phi(z)) = 1 + \sum_{n=1}^{\infty} q_n z^n$ satisfies the condition $\Re\, q(z) > 0$, it follows that $|q_n| \leq 2$. From the inequality $|q_k| \leq 2$ $(k = 1, 2, \cdots, n)$, we deduce that $|a_n| \leq n$. The inequality $|a_n| \leq n$ for the coefficients of functions subordinate to $g(z) = z/(1 - z)^2$ (or, equivalently, functions normalized by $f(0) = 0$ and not assuming the real numbers $w \leq -1/4$ in Δ) was also proved in [13, p. 493] by Littlewood and in [23, p. 65] by Rogosinski. We now see that this simple situation essentially proves the inequality for functions subordinate to some function in St.

Similar observations can be made about other results contained in [23]. We call attention to a related paper [19] by Robertson, who proved that the inequality $|a_n| \leq n$ holds for the coefficients of a function subordinate to some function in C.

Robertson [20] introduced the concept of quasi-subordination, which generalizes both majorization and subordination. If f and g are analytic in Δ, then f is quasi-subordinate to g in Δ if there exist two analytic functions ϕ and ω such that $|\phi(z)| \leq 1$, $|\omega(z)| < 1$, $\omega(0) = 0$, and $f(z) = \phi(z)\, g(\omega(z))$. Our earlier arguments are also applicable to this situation. We find that in order to solve a linear extremal problem over the class $\{f\}$, we may assume in the relation above between f and g that g is an extreme point (at least when the family $\{g\}$ is compact). In [20], Robertson proved several coefficient inequalities associated with quasi-subordination. To these results we can now add that our Theorem 1 holds where the relation between f and g is quasi-subordination. The proof needs only be given for g in $\mathfrak{E}(\mathfrak{H}\,C)$. With this simplification, we find that the argument in the proof of Theorem 1 again shows that $|B_{2n-1}| \leq 1$. To carry out this argument, we must take advantage of Theorems 3 and 9 in [20] as used there by Robertson to prove Theorem 11.

There are other indications that this approach has a wide range of applications. In his dissertation, David Hallenbeck considers some problems in this direction. For example, he shows that if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is subordinate to g in Δ and if $g \in St$ and g is odd, then $|a_n| \leq 1$ for $n = 1, 2, \cdots$. Besides the direct coefficient problems mentioned here, applications of this method should be of general consequence, since J can be any continuous linear functional.

3. ESTIMATES ON L^p-MEANS OF UNIVALENT FUNCTIONS

Corresponding to each analytic function in Δ, we let

$$J(f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta, \qquad \text{where } 0 < r < 1 \text{ and } p > 0.$$

We shall be interested in maximizing J(f) over various families of functions univalent in Δ. The problems we solve depend on our knowledge of the extreme points for the family. Our method is generally applicable, at least as an initial simplification of such problems. It will be more convenient to consider the norm

$$\|f\| = \left\{\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta\right\}^{1/p},$$

instead of J(f).

Suppose that $\mathscr{F}$ is a compact subset of $\mathscr{A}$ (so that $\mathscr{F}$ is closed and locally uniformly bounded). Then $\mathfrak{H}\mathscr{F}$ is compact and $M = \max_{f \in \mathfrak{H}\mathscr{F}} \|f\|$ exists, since $\|f\|$ is a continuous functional. The set

$$\mathscr{F}_0 = \{f\colon f \in \mathfrak{H}\mathscr{F} \text{ and } \|f\| = M\}$$

is an extremal subset of $\mathfrak{H}\mathscr{F}$; that is, if $tg + (1 - t)h \in \mathscr{F}_0$, where $0 < t < 1$ and g and h belong to $\mathfrak{H}\mathscr{F}$, then $g \in \mathscr{F}_0$ and $h \in \mathscr{F}_0$. This follows from the Minkowski inequality, if we assume that $p \geq 1$. Indeed,

$$M = \|tg + (1 - t)h\| \leq t\|g\| + (1 - t)\|h\| \leq tM + (1 - t)M = M,$$

and thus $\|g\| = \|h\| = M$. Since $\mathscr{F}_0$ is compact and nonvoid, $\mathscr{F}_0$ must have an extremal point, and such a point belongs to $\mathfrak{E}(\mathfrak{H}\mathscr{F})$, since $\mathscr{F}_0$ is an extremal set. Further, the compactness of $\mathscr{F}$ implies that $\mathfrak{E}(\mathfrak{H}\mathscr{F}) \subset \mathscr{F}$. These relations show that if $p \geq 1$, then

$$\max_{f \in \mathscr{F}} \|f\| = \max_{f \in \mathfrak{H}\mathscr{F}} \|f\| = \max_{f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})} \|f\|.$$

The argument above can be reproduced if we replace $\|f\|$ by $\|f^{(n)}\|$, where $f^{(n)}$ denotes the nth derivative of f. We simply point out that $\|f^{(n)}\|$ is a continuous functional and the nth derivative of $tg + (1 - t)h$ is $tg^{(n)} + (1 - t)h^{(n)}$. Therefore, if $\mathscr{F}$ is a compact subset of $\mathscr{A}$ and $p \geq 1$, then

$$\max_{f \in \mathscr{F}} \|f^{(n)}\| = \max_{f \in \mathfrak{H}\mathscr{F}} \|f^{(n)}\| = \max_{f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})} \|f^{(n)}\| \quad \text{for } n = 1, 2, \cdots.$$

A refinement of this can be made if the functions in $\mathscr{F}$ satisfy some additional conditions. Suppose that each function f in $\mathscr{F}$ is normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. If $f \in \mathfrak{H}\mathscr{F}$ and $f \notin \mathfrak{E}(\mathfrak{H}\mathscr{F})$, then $f = tg + (1 - t)h$, where $0 < t < 1$, $g \neq h$, $g \in \mathfrak{H}\mathscr{F}$, and $h \in \mathfrak{H}\mathscr{F}$. For $p \geq 1$, Minkowski's inequality shows that

$$\|f\| \leq t\|g\| + (1 - t)\|h\| \leq \max(\|g\|, \|h\|).$$

Equality in the Minkowski inequality occurs only when $tg = a(1 - t)h$ almost everywhere, where a is real and $a \neq 0$. Since g and h are continuous in θ, this would imply that $tg(re^{i\theta}) = a(1 - t)h(re^{i\theta})$ for all θ; that is, $tg(z) = a(1 - t)h(z)$ for all z with $|z| = r$. By the analyticity of g and h, this implies that $tg = a(1 - t)h$. Because of the normalization $g'(0) = h'(0) = 1$, we see that $t = a(1 - t)$, and since $a \neq 0$ and $1 - t \neq 0$, we conclude that $g = h$. Since $g \neq h$, this shows that equality in the Minkowski inequality does not occur, and therefore $\|f\| < \max(\|g\|, \|h\|)$. Thus, the only functions f_0 in $\mathfrak{H}\mathscr{F}$ for which $\|f_0\| = \max_{f \in \mathfrak{H}\mathscr{F}} \|f\|$ are extreme points of $\mathfrak{H}\mathscr{F}$. Using both normalizations $f(0) = 0$ and $f'(0) = 1$, we can likewise conclude that the extreme points of $\mathfrak{H}\mathscr{F}$ are the only functions f_0 in $\mathfrak{H}\mathscr{F}$ for which $\|f_0'\| = \max_{f \in \mathfrak{H}\mathscr{F}} \|f'\|$.

We have already indicated an important simplification in solving the problems in this section. We shall also need an integral inequality concerning the symmetrically decreasing rearrangement of functions, as discussed in [7, p. 278]. In general, let ϕ^* denote the symmetrically decreasing rearrangement of ϕ. Then, for all nonnegative, integrable functions ϕ and ψ on $[-c, c]$ $(c > 0)$,

$$(1) \qquad \int_{-c}^{c} \phi(x)\psi(x)\,dx \leq \int_{-c}^{c} \phi^*(x)\psi^*(x)\,dx.$$

The proof of this inequality is given by J. Clunie and P. L. Duren in [4] for the more general situation where there are three functions in the integrands. The inequality is used by Clunie and Duren to show that for each function f in C,

$$\frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|\,d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |k'(re^{i\theta})|\,d\theta,$$

where $k(z) = z/(1 - z)^2$ is the Koebe function. Earlier, Duren [6] had proved this inequality for a special subclass of C. Theorem 2 includes the inequality as a special case.

Our use of (1) involves functions with certain properties of symmetry. If ϕ is continuous and even on $[-c, c]$, and if ϕ is decreasing on $[0, c]$, then $\phi^* = \phi$. Also, if ψ has the form $\psi(x) = \eta(x + \alpha)$, where η is continuous for all x, decreases on $[0, c]$, and is periodic with period 2c, then $\psi^* = \eta$ on $[-c, c]$.

THEOREM 2. *Let* f *be analytic and close-to-convex in* Δ *and normalized by* $f(0) = 0$ *and* $f'(0) = 1$. *Suppose that* $0 < r < 1$ *and* $k(z) = z/(1-z)^2$. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |k(re^{i\theta})|^p\, d\theta \qquad (p = 1, 2, \cdots),$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |k^{(n)}(re^{i\theta})|^p\, d\theta \qquad (n = 1, 2, \cdots;\ p \ge 1).$$

Proof. Let $I(f) = \frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p\, d\theta$, where $n = 0, 1, 2, \cdots$ and $f^{(0)}$ denotes f. We are to prove that $I(f) \le I(k)$, and since $p \ge 1$, our introductory remarks reduce the problem to a consideration of the functions f in $\mathfrak{E}(\mathfrak{H}C)$. The set $\mathfrak{E}(\mathfrak{H}C)$ consists of the functions of the form

$$f(z) = \left[z - \frac{1}{2}(\varepsilon + \delta)z^2 \right] / [1 - \delta z]^2, \qquad \text{where } |\varepsilon| = |\delta| = 1 \text{ and } \varepsilon \ne \delta.$$

If η is a complex number ($|\eta| = 1$) and $g(z) = f(\eta z)/\eta$, then $|g^{(n)}(z)| = |f^{(n)}(\eta z)|$ for $n = 0, 1, 2, \cdots$, and thus $I(g) = I(f)$. If we set $\eta = 1/\delta$ and write $a = \varepsilon/\delta$, it follows that we need only prove the inequality $I(f) \le I(k)$ for the functions

$$f(z) = \left[z - \frac{1}{2}(a + 1)z^2 \right] / [1 - z]^2, \qquad \text{where } |a| = 1 \text{ and } a \ne 1.$$

Also, we shall let $b = (a + 1)/2$, so that $|b - 1/2| = 1/2$.

First we consider the case $n = 0$, and we begin by showing that if $|b - 1/2| \le 1/2$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1 - bz}{1 - z}\right|^{2p} d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1 - z|^{2p}}\, d\theta,$$

where $z = re^{i\theta}$ and $p = 1, 2, \cdots$. If $(1 - bz)/(1 - z) = \sum_{n=0}^{\infty} b_n z^n$ for $|z| < 1$, then $b_0 = 1$ and $b_n = 1 - b$ for $n \ge 1$. Because $|b - 1/2| \le 1/2$, it follows that $|b_n| \le 1$ for $n \ge 1$. Also, $|b_n| = 1$ only for $b = 0$, and then $b_k = 1$ for $k = 1, 2, \cdots$. This implies that the coefficients of the power series

$$\left(\frac{1 - bz}{1 - z}\right)^p = 1 + \sum_{n=1}^{\infty} c_n^{(p)}(b)\, z^n$$

satisfy the inequality $|c_n^{(p)}(b)| \le c_n^{(p)}(0)$ for $n \ge 1$ and $p = 1, 2, \cdots$. Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1-bz}{1-z}\right|^{2p}d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\left(\frac{1-bz}{1-z}\right)^p\right|^2 d\theta = 1+\sum_{n=1}^{\infty}|c_n^{(p)}(b)|^2 r^{2n}$$

$$\leq 1+\sum_{n=1}^{\infty}[c_n^{(p)}(0)]^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-z|^{2p}}d\theta .$$

Let $f(z) = (z - bz^2)/(1 - z)^2$, where $|b - 1/2| = 1/2$. Then, by the Cauchy-Schwarz inequality and the result above, we find that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(z)}{z}\right|^p d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1-bz}{(1-z)^2}\right|^p d\theta$$

$$\leq \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1-bz}{1-z}\right|^{2p}d\theta \cdot \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-z|^{2p}}d\theta\right\}^{1/2}$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-z|^{2p}}d\theta .$$

This is the same as the inequality $I(f) \leq I(k)$, and it proves the theorem for $n = 0$ when $p = 1, 2, \cdots$. (We thank D. R. Wilken for pointing out the applicability of the Cauchy-Schwarz inequality. For $n = 0$ and $p = 1$, we had first obtained our theorem by using the lemma.)

Next consider the case $n = 1$. If $f(z) = \left[z - \frac{1}{2}(a+1)z^2\right]/[1 - z]^2$, then $f'(z) = (1 - az)/(1 - z)^3$, and thus, for $a = e^{i\alpha}$,

$$I(f) = \frac{1}{2\pi}\int_0^{2\pi}\frac{|1-e^{i\alpha}re^{i\theta}|^p}{|1-re^{i\theta}|^{3p}}d\theta . \tag{2}$$

In this integral we may consider the interval of integration to be $[-\pi, \pi]$. Then the symmetrically decreasing rearrangement of $\phi(\theta) = |1 - re^{i\theta}|^{-3p}$ is ϕ itself, and if $\psi(\theta) = |1 - e^{i\alpha}re^{i\theta}|^p$, then $\psi^*(\theta) = |1 + re^{i\theta}|^p$. If we apply (1) to (2), it shows that

$$I(f) \leq \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1+re^{i\theta}}{(1-re^{i\theta})^3}\right|^p d\theta = I(k),$$

and this proves the theorem for $n = 1$.

The higher derivatives of f can be treated similarly. First we note that if $f(z) = \left[z - \frac{1}{2}(a+1)z^2\right]/[1 - z]^2$, then we can show inductively that

$$f^{(n)}(z) = [\alpha - \beta a - \gamma az]/[1 - z]^{n+2} \qquad \text{for } n = 1, 2, \cdots,$$

where $\alpha = \frac{1}{2}[(n+1)!]$, $\beta = \frac{1}{2}[n!(n-1)]$, and $\gamma = n!$. Thus, for this function

$$I(f) = \frac{1}{2\pi} \int_0^{2\pi} \frac{|\alpha - \beta a - \gamma a r e^{i\theta}|^p}{|1 - re^{i\theta}|^{(n+2)p}} \, d\theta \,. \tag{3}$$

We may assume that the interval of integration is $[-\pi, \pi]$. With varying θ, the expression $|\alpha - \beta a - \gamma a r e^{i\theta}|$ gives the distance between the point $\alpha - \beta a$ and the points on the circle centered at the origin and with radius γr. Therefore, the symmetrically decreasing rearrangement of $\psi(\theta) = |\alpha - \beta a - \gamma a r e^{i\theta}|^p$ is $\psi^*(\theta) = |d + \gamma r e^{i\theta}|^p$, where $d = |\alpha - \beta a|$. If $\phi(\theta) = |1 - re^{i\theta}|^{-(n+2)p}$, then $\phi^*(\theta) = \phi(\theta)$. Therefore, because of (1) and (3), we conclude that

$$I(f) \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{d + \gamma z}{(1-z)^{n+2}} \right|^p d\theta \,. \tag{4}$$

Since $0 \leq d \leq \alpha + \beta = n\gamma$, we may write $d = (\cos \omega) n\gamma = \frac{1}{2}(e^{i\omega} + e^{-i\omega}) n\gamma$ $(0 \leq \omega \leq \pi/2)$, and thus

$$\frac{d + \gamma z}{(1-z)^{n+2}} = \frac{\gamma}{2} \frac{n e^{i\omega} + z}{(1-z)^{n+2}} + \frac{\gamma}{2} \frac{n e^{-i\omega} + z}{(1-z)^{n+2}} \,.$$

An application of the Minkowski inequality to this relation shows that

$$\left\{ \int_0^{2\pi} \left| \frac{d + \gamma z}{(1-z)^{n+2}} \right|^p d\theta \right\}^{1/p}$$

$$\leq \frac{\gamma}{2} \left\{ \int_0^{2\pi} \left| \frac{n e^{i\omega} + z}{(1-z)^{n+2}} \right|^p d\theta \right\}^{1/p} + \frac{\gamma}{2} \left\{ \int_0^{2\pi} \left| \frac{n e^{-i\omega} + z}{(1-z)^{n+2}} \right|^p d\theta \right\}^{1/p}$$

$$= \frac{\gamma}{2} \left\{ \int_0^{2\pi} \left| \frac{n + e^{-i\omega} z}{(1-z)^{n+2}} \right|^p d\theta \right\}^{1/p} + \frac{\gamma}{2} \left\{ \int_0^{2\pi} \left| \frac{n + e^{i\omega} z}{(1-z)^{n+2}} \right|^p d\theta \right\}^{1/p} .$$

The symmetrically decreasing rearrangement of each of the functions $|n + e^{-i\omega} re^{i\theta}|^p$ and $|n + e^{i\omega} re^{i\theta}|^p$ is the function $|n + re^{i\theta}|^p$. Applying (1) to each of the last two integrals, and using the relation $n\gamma = \alpha + \beta$, we conclude that

$$\left\{ \int_0^{2\pi} \left| \frac{d + \gamma z}{(1-z)^{n+2}} \right|^p d\theta \right\}^{1/p} \leq \gamma \left\{ \int_0^{2\pi} \left| \frac{n + z}{(1-z)^{n+2}} \right|^p d\theta \right\}^{1/p}$$

$$= \left\{ \int_0^{2\pi} \left| \frac{\alpha + \beta + \gamma z}{(1-z)^{n+2}} \right|^p d\theta \right\}^{1/p} .$$

Because the function k corresponds to the case $a = -1$, this inequality and (4) show that

$$I(f) \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\alpha + \beta + \gamma z}{(1 - z)^{n+2}} \right|^p d\theta = I(k).$$

This completes the proof of the theorem.

As we pointed out earlier, the result of Theorem 2 was proved by Clunie and Duren for the case where $n = 1$ and $p = 1$. If $n = 1$ and p is an even integer, it can also be deduced from the inequalities $|a_k| \leq k$ $(k = 1, 2, \cdots)$ on the coefficients of a function in C (proved by M. O. Reade in [17]). The argument depends on the formula

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} |b_k|^2 r^{2k},$$

valid for a function $g(z) = \sum_{k=0}^{\infty} b_k z^k$ analytic in Δ. The question whether equality in the theorem occurs only for the functions $f(z) = z/(1 - \varepsilon z)^2$, where $|\varepsilon| = 1$, remains unsettled except for the case $n = 0$ and some of the special cases just mentioned.

We expect that when $n = 0$, the inequality $I(f) \leq I(k)$ holds for all $p \geq 1$. This conjecture would be settled if we knew that if $|b - 1/2| = 1/2$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1 - bz}{(1 - z)^2} \right|^p d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - z|^{2p}} d\theta \quad \text{for } p \geq 1.$$

As we showed in the proof of Theorem 2, the integral inequality above would follow if the coefficients of the power series for $[(1 - bz)/(1 - z)]^p$ satisfied the inequality $|c_n^{(p)}(b)| \leq c_n^{(p)}(0)$ for $n = 1, 2, \cdots$. More generally, it seems likely that the inequality $I(f) \leq I(k)$ holds for $n = 0, 1, 2, \cdots$ simply if $p > 0$. Our method cannot deal with the situation where $0 < p < 1$. On the other hand, when $p \geq 1$, our proof actually yields a more general result, namely, that $I(f) \leq I(k)$ for all functions f in $\mathfrak{H}C$.

The right-hand side of the inequality in Theorem 2 can be explicitly computed, in some cases. For example, when $n = 0$ and $p = 1$, then

$$\frac{1}{2\pi} \int_0^{2\pi} |k(re^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{r}{|1 - re^{i\theta}|^2} d\theta = r\{1 + r^2 + r^4 + \cdots\} = \frac{r}{1 - r^2},$$

and thus $\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta \leq \frac{r}{1 - r^2}$ for each f in C.

Our arguments are applicable to a number of similar situations. For example, since the set of functions $z/(1 - \varepsilon z)$ $(|\varepsilon| = 1)$ is precisely $\mathfrak{E}(\mathfrak{H}K)$, we find that if $f \in K$, then

$$\frac{1}{2\pi} \int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |h^{(n)}(re^{i\theta})|^p d\theta \tag{5}$$

for $n = 0, 1, 2, \cdots$ and $p \geq 1$, where $h(z) = z/(1 - z)$. The proof is easier than the proof of Theorem 2, because the integral $I(f)$ is constant on $\mathfrak{E}(\mathfrak{H} K)$. For $n = 1$ and $p = 1$, the result was proved by A. Marx in [15] and later by F. R. Keogh in [11]. In this case, the right-hand side of (5) is explicitly computable, and it yields the inequality

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|\, d\theta \leq \frac{1}{1 - r^2},$$

or $L(r) \leq 2\pi r/(1 - r^2)$, where $L(r)$ denotes the length of the image of the circle $|z| = r$ under the mapping $z \to f(z)$.

Inequality (5) actually holds for all $p > 0$, when $n = 0$ or $n = 1$. For $n = 0$, this was shown by Robertson in [18]. His argument depends on the two facts that if $f \in K$, then $f(z)/z$ is subordinate to $1/(1 - z)$ in Δ, and that if g is subordinate to G in Δ, then

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})|^p\, d\theta \qquad \text{for } p > 0.$$

The last inequality was proved by J. E. Littlewood in [13] (see also [9, p. 422]). When $n = 1$, the inequality (5) holds for all $p > 0$, because of Littlewood's inequality and the result (proved by Marx and Strohhäcker in [15] and [24]) that if $f \in K$, then $f'(z)$ is subordinate to $1/(1 - z)^2$ in Δ. The result that $f(z)/z$ is subordinate to $1/(1 - z)$ for functions in K was also shown in [15] and [24]. More recently, a short and elegant proof of this was given by T. J. Suffridge in [25].

The inequality

$$\frac{1}{2\pi} \int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |k^{(n)}(re^{i\theta})|^p\, d\theta \tag{6}$$

holds for $n = 0, 1, 2, \cdots$ and $p \geq 1$ for each function in St. For $n \geq 1$, inequality (6) is contained in the more general result of Theorem 2. However, we point out that the argument for the smaller class St is quite simple, not requiring the details for C as given in the proof of Theorem 2. In the case $n = 0$, inequality (6) holds for all $p > 0$, as was shown by Robertson in [18] (see also [11] for the case $n = 0$, $p = 1$). For $n = 1$ and $p = 1$, the inequality was obtained in [15] by Marx.

Let $\mathscr{P}$ denote the family of analytic functions f in Δ that satisfy the conditions $\Re f(z) > 0$ and $f(0) = 1$. It follows from the Herglotz representation of functions in $\mathscr{P}$ that the extreme points of $\mathscr{P}$ are precisely the functions $f(z) = (1 + \varepsilon z)/(1 - \varepsilon z)$ with $|\varepsilon| = 1$. The arguments presented earlier may be applied to this family. As a consequence, we can prove that

$$\frac{1}{2\pi} \int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f_0^{(n)}(re^{i\theta})|^p\, d\theta \tag{7}$$

$$\text{for } n = 0, 1, 2, \cdots \text{ and } p \geq 1,$$

where $f_0(z) = (1 + z)/(1 - z)$. Part of the demonstration of the inequality (7) is quite simple, since the integral $I(f)$ is constant on $\mathfrak{E}(\mathscr{P})$. Apparently, the inequality is new except for some special cases of n and p. For $n = 0$ it follows by Littlewood's inequality since f is subordinate to f_0, and it even holds for all $p > 0$. For $n = 1$ and $p = 1$, (7) was first proved by Rogosinski in [22] (see also [16]). Again the case where p is an even integer is simple, and a proof of this can be based on the inequalities $|a_n| \leq 2$ for the coefficients of a function in $\mathscr{P}$.

These questions can also be resolved for the class R, or, more generally, for the class T. We need only point out that $T = \mathfrak{H}R \subset \mathfrak{H}St$ [3, p. 106], and that $I(f) \leq I(k)$, in particular, for all functions f in $\mathfrak{H}St$, if $n = 0, 1, 2, \cdots$ and $p \geq 1$. The special cases of this, corresponding to $n = 0$ and $p = 1$ and to $n = 1$ and $p = 1$, were treated by Robertson in [18].

We may use the argument given at the beginning of this section to deduce that

$$\max_{f \in \mathscr{F}} \|f\| = \max_{f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})} \|f\|$$

in a more general situation. This was pointed out by L. Brickman. Namely, if J is a subadditive, continuous functional on $\mathscr{A}$, and if $\mathscr{F}$ is compact, then

$$\max_{f \in \mathscr{F}} |J(f)| = \max_{f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})} |J(f)| .$$

(J is subadditive if it is real-valued and $J(tg + (1 - t)h) \leq tJ(g) + (1 - t)J(h)$ for each real number t $(0 < t < 1)$ and for all functions g and h in $\mathscr{A}$.)

Another possible direction for applications of extreme-point theory is based on a generalization of our considerations, obtained by replacing differentiation by a linear operator. Specifically, if $\mathscr{L}$ is a continuous linear operator on $\mathscr{A}$ to $\mathscr{A}$, then

$$\max_{f \in \mathscr{F}} \|\mathscr{L}(f)\| = \max_{f \in \mathfrak{E}(\mathfrak{H}\mathscr{F})} \|\mathscr{L}(f)\| ,$$

where $\mathscr{F}$ is a compact subset of $\mathscr{A}$. For example, suppose that $\mathscr{L}$ is the nth partial sum of the power series for f. We can assert that

$$\frac{1}{2\pi} \int_0^{2\pi} |g(z)|^p \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |g_0(z)|^p \, d\theta \qquad \text{for } p \geq 1,$$

where g_0 is the nth partial sum of the power series of some function in $\mathfrak{E}(\mathfrak{H}\mathscr{F})$. When $p = 2$, this shows that in order to maximize $\sum_{k=0}^{n} |a_k|^2 r^{2k}$, we need only consider extreme points f. Our arguments also show that the expression $\sum_{k=0}^{n} |a_k|^2$ is maximized over $\mathscr{F}$ if it is maximized merely over $\mathfrak{E}(\mathfrak{H}\mathscr{F})$. In [21], R. M. Robinson proves some interesting results concerning linear operators somewhat related to our considerations.

Our use of extreme-point theory can be combined nicely with Littlewood's inequality concerning subordination. For example, suppose that $g \in St$ and that f is quasi-subordinate to g in Δ. Writing $g(z) = \phi(z) f(\omega(z))$, where ϕ and ω have their usual meaning, we see that if $p \geq 1$, then

$$\frac{1}{2\pi}\int_0^{2\pi} |g(z)|^p\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f(\omega(z))|^p\, d\theta$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(z)|^p\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |k(z)|^p\, d\theta\,,$$

where $k(z) = z/(1 - z)^2$. The first two inequalities were pointed out in [20] by Robertson, and the third depends on our considerations of extreme-point theory.

A number of comments we have made are valid for the class S. The conclusions we thereby obtain assert that in order to maximize various quantities over S, it suffices to consider the functions in $\mathfrak{E}(\mathfrak{H}S)$. At this point, we know of no examples where this becomes an effective step in solving such a problem. In part, this is due to our lack of information about $\mathfrak{E}(\mathfrak{H}S)$. Of course, the set $\mathfrak{E}(\mathfrak{H}S)$ can be expected to be much more complicated than the simple sets $\mathfrak{E}(\mathfrak{H}St)$, $\mathfrak{E}(\mathfrak{H}R)$, $\mathfrak{E}(\mathfrak{H}C)$, $\mathfrak{E}(\mathfrak{H}K)$, or $\mathfrak{E}(\mathscr{P})$, which are so precisely determined in terms of one or two parameters. One of the few facts known about $\mathfrak{E}(\mathfrak{H}S)$ was proved by L. Brickman in [2]; namely, if $f \in \mathfrak{E}(\mathfrak{H}S)$, there the complement of $f(\Delta)$ is a continuous curve tending to infinity with increasing modulus.

Added June 4, 1972. We thank Professor Paul Eenigenburg for calling our attention to the fact that the results concerning the family of functions with positive real part pointed out in our remarks after the proof of Theorem 2 were proved earlier by Y. Komatu in the paper *On analytic functions with positive real part in a circle,* Kōdai Math. Sem. Rep. 10 (1958), 64-83.

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State University of New York
Albany, New York 12203