HOLOMORPHIC FUNCTIONS WITH LINEARLY ACCESSIBLE ASYMPTOTIC VALUES

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Let D and C denote the open unit disc and the unit circle. An arc $T \subset D$ *ends* at $z_0 \in C$ if $T \cup z_0$ is a Jordan arc. A holomorphic function f in D has *asymptotic value* w_0 at $z_0 \in C$ if there exists an arc $T \subset D$, ending at z_0, such that $f(z) \to w_0$ as $z \to z_0$ $(z \in T)$. The arc T is then an *asymptotic path* of f. If f maps T one-to-one onto a linear segment ending at w_0, then f has a *linearly accessible* asymptotic value at z_0. Let $A_L(f)$ denote the set of points at which f has linearly accessible values. G. R. MacLane [8, Theorems 3, 5, 7] has given several sufficient conditions for $A_L(f)$ to be dense on C. We shall give a necessary and sufficient condition for $A_L(f)$ to be dense on C.

Let S be a nonempty subset of D. For each r $(0 < r < 1)$, let the components of $S \cap \{z: r < |z| < 1\}$ be $S_\beta(r)$ $(\beta \in B)$. Let $d_\beta(r)$ be the diameter of $S_\beta(r)$, and let $d(r) = \sup_{\beta \in B} d_\beta(r)$. Clearly, d is a nonincreasing function of r. The set S *ends at points* of C if $d(r) \downarrow 0$ as $r \uparrow 1$.

If $w = f(z)$ is a nonconstant, holomorphic function in D, we denote by F the Riemann surface of f^{-1} (as a covering surface over the w-plane). Let p denote the projection from F onto the w-plane, and let $\tilde{f}$ be the one-to-one conformal map of D onto F, so that $f = p \circ \tilde{f}$. Corresponding to each set S in the w-plane, we denote by F_S the set of points of F lying over S.

MacLane's class $\mathscr{A}$ is the class of nonconstant holomorphic functions in D *that have asymptotic values at a dense subset of* C. *A function* f *belongs to class $\mathscr{L}$ if it is nonconstant and holomorphic in* D *and if for each* $r \geq 0$ *the level set* $\{z: |f(z)| = r\}$ *ends at points of* C. MacLane [7, Theorem 1] proved that $\mathscr{A} = \mathscr{L}$. We now state our main result.

THEOREM 1. *Let* f *be a nonconstant, holomorphic function in* D. *A necessary and sufficient condition for* $A_L(f)$ *to be dense on* C *is that there exists a line* K *in the* w*-plane such that the set* $\tilde{f}^{-1}(F_K)$ *ends at points of* C.

REMARKS. 1. In the notation of this paper, we can restate the assertion $\mathscr{A} = \mathscr{L}$ as follows. A necessary and sufficient condition for a nonconstant holomorphic function f to belong to class $\mathscr{A}$ is that the set $\tilde{f}^{-1}(F_{|w|=r})$ ends at points of C, for each $r \geq 0$. From this restatement it is clear that the condition of Theorem 1 for lifting lines is analogous to MacLane's condition expressed in $\mathscr{A} = \mathscr{L}$ for lifting circles.

2. In proving Theorem 1, we shall prove that a necessary condition for $A_L(f)$ to be dense on C is that the set $\tilde{f}^{-1}(F_K)$ ends at points of C *for every line* K *in the* w*-plane. Hence, if the set* $\tilde{f}^{-1}(F_K)$ *ends at points of* C *for one line* K *in the* w*-plane, then* $A_L(f)$ *is dense on* C *and hence the set* $\tilde{f}^{-1}(F_K)$ *ends at points for every line* K *in the* w*-plane.*

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3. The condition of Theorem 1 is obviously a sufficient condition for a holomorphic function in D to belong to class $\mathscr{A}$. To see that it is not a necessary condition, consider K. Barth and W. Schneider's example [2, Main Theorem] of a holomorphic function f such that $f \in \mathscr{A}$ but $\exp f \notin \mathscr{A}$. If there were a line K in the w-plane such that the set $\tilde{f}^{-1}(F_K)$ ends at points of C, then by the preceding remark the set $\tilde{f}^{-1}(F_K)$ would end at points of C for every line K. Hence, for each $r \geq 0$, the level set $\{z: |\exp f(z)| = r\}$ would end at points of C. Therefore, exp f would belong to class $\mathscr{L}$ and hence to class $\mathscr{A}$.

It is convenient to introduce the following notation. If a and b are two real numbers $(0 \leq a \leq b \leq a + 2\pi)$, we write $C(a, b) = \{e^{i\theta}: a < \theta < b\}$. For each subset S of the complex plane, we denote by $\bar{S}$ and ∂S the closure and the boundary of S in the Euclidean topology of the plane. Finally, a sequence $\{T_k\}$ of curves lying in D *tends to* the arc C(a, b) if for each $\varepsilon > 0$, there exists a positive integer n such that for $k \geq n$

(i) $T_k \subset \{z: 1 - \varepsilon < |z| < 1\}$,

(ii) $|\inf\{\arg z: z \in T_k\} - a| < \varepsilon$,

(iii) $|\sup\{\arg z: z \in T_k\} - b| < \varepsilon$.

We are now ready to prove Theorem 1.

Proof of the sufficiency. We shall do most of our work in the following two lemmas.

LEMMA 1. *Suppose that no point of the arc* C(a, b) *on* C *is the end of an asymptotic path for a linearly accessible asymptotic value, that* K *is a line such that the set* $\tilde{f}^{-1}(F_K)$ *ends at points of* C, *and that* z_0 *is a point of* D *such that* $f(z_0) \in K$ *and* $f'(z_0) \neq 0$. *Then* z_0 *lies on a crosscut* T *of* D *such that* $f(T) \subset K$, *the endpoints of* T *lie in* C - C(a, b), *and* f *has a linearly accessible asymptotic value along each end of* T.

Proof. Let (s, t) be a maximal segment of the line K (in the w-plane) that can be lifted into the surface F so that the lifted segment contains the point $\tilde{f}(z_0)$. If the surface F has no branchpoints over K, there exists exactly one such segment. If F has branchpoints over K, these branchpoints are at most countable, and the segment (s, t) is uniquely determined if for each branchpoint we determine in advance the sheet on which we shall exit from the branchpoint, in case we encounter it. To see how this could be done, let $q_1, q_2, \cdots$ be the branchpoints of F, and let n_k be the order of the branchpoint q_k. Then there exists a neighborhood N_k of $p(q_k)$, the projection of q_k onto the w-plane, such that the component of $F \cap p^{-1}(N_k)$ containing q_k has precisely $n_k + 1$ sheets $S_1^k, S_2^k, \cdots, S_{n_k+1}^k$ lying over $N_k - \{p(q_k)\}$. If we encounter q_k in the lifting process, then we agree in advance to exit in the sheet S_1^k.

Let G denote the lifted segment in F, and write $T = \tilde{f}^{-1}(G)$. Clearly, s and t are linearly accessible asymptotic values of f along the two ends of T, and our assumption on C(a, b) implies that T ends at points of C - C(a, b).

In the following lemma, let $\zeta = \exp[i(a + b)/2]$.

LEMMA 2. *Under the conditions of Lemma* 1, *the set* $\tilde{f}^{-1}(F_K) \cap \{|z - \zeta| < r\}$ *is empty for all sufficiently small values* r.

Proof. Suppose $\tilde{f}^{-1}(F_K)$ meets each neighborhood of ζ. Since $f'(z) = 0$ for at most countably many z, there exists a point $z_1 \in \tilde{f}^{-1}(F_K)$ such that $|z_1 - \zeta| < 1/2$

and $f'(z_1) \neq 0$. By Lemma 1, some arc T_1 of $\tilde{f}^{-1}(F_K)$ contains z_1 and ends at points of $C - C(a, b)$. In the component of $D - T_1$ whose boundary contains the arc $C(a, b)$, there exists a point $z_2 \in f^{-1}(F_K)$ such that $|z_2 - \zeta| < 2^{-2}$ and $f'(z_2) \neq 0$. By Lemma 1, some arc T_2 of $\tilde{f}^{-1}(F_K)$ contains z_2 and ends at points of $C - C(a, b)$. Proceeding inductively, we arrive at a sequence of arcs T_n and points z_n such that

(i) $z_n \in T_n \subset \tilde{f}^{-1}(F_K)$,

(ii) T_n ends at points of $C - C(a, b)$,

(iii) $|z_n - \zeta| < 2^{-n}$.

Clearly, a subsequence of $\{T_n\}$ must tend to an arc containing either the arc $C\left(a, \frac{a+b}{2}\right)$ or the arc $C\left(\frac{a+b}{2}, b\right)$. This contradicts the hypothesis that the set $\tilde{f}^{-1}(F_K)$ ends at points of C, and establishes Lemma 2.

To finish the proof of the sufficiency of the condition of Theorem 1, suppose $A_L(f)$ is not dense on C and $C(a, b)$ is an arc having no point that is an end of an asymptotic path for a linearly accessible asymptotic value. Let ζ be the midpoint of $C(a, b)$, and let K be a line in the w-plane such that $\tilde{f}^{-1}(F_K)$ ends at points of C. By Lemma 2, the set $\tilde{f}^{-1}(F_K) \cap \{|z - \zeta| < r_0\}$ is empty for some r_0. Hence, the values $w = f(z)$ for $z \in D \cap \{|z - \zeta| < r_0\}$ lie entirely in one of the half-planes with boundary K. Let J denote the arc $C \cap \{|z - \zeta| < r_0\}$. Then, by an easy extension of Fatou's well-known theorem on radial limits [4, p. 17], f has radial limits almost everywhere on J; hence, by a theorem of MacLane [8, Theorem 5], f has linearly accessible asymptotic values on a dense subset of J. Since $J \subset C(a, b)$, this contradicts the choice of $C(a, b)$ and completes the proof of the sufficiency.

Proof of the necessity. We shall use a lifting technique of Barth and Schneider [3]. Suppose there is a line K in the w-plane such that the set $\tilde{f}^{-1}(F_K)$ does not end at points of C. Then there exists a sequence $\{T_n\}$ of mutually disjoint arcs tending to an arc $C(a, b)$ on C such that $f(T_n) \subset K$. By translating and rotating the w-plane, if necessary, we can assume that K is the real axis. Since f is an open map, there exists a sequence $\{z_n\}$ of points in D such that $z_n \to \exp[i(a+b)/2]$ and $\Im(f(z_n)) > 0$ for each n.

Let $[f(z_n), s_n)$ be a maximal half-open segment of the line

$$J_n = \{w\colon \Im(w) = \Im(f(z_n)),\ \Re(w) \geq \Re(f(z_n))\}$$

that can be lifted into the surface F so that $\tilde{f}(z_n)$ is the initial point of the lifted segment. Let G_n denote the lifted segment in F, and R_n the arc $\tilde{f}^{-1}(G_n)$ in D. Clearly, the set $R_n \cap T_k$ is empty for all n and k, and s_n is a linearly accessible asymptotic value of f along R_n. Hence, there exists a sequence $\{R_k^*\}$ of arcs such that R_k^* is a subarc of some R_n and $\{R_k^*\}$ tends to one of the two arcs $C\left(a, \frac{a+b}{2}\right)$ and $C\left(\frac{a+b}{2}, b\right)$, say $C\left(a, \frac{a+b}{2}\right)$.

Since $A_L(f)$ is dense, f has a linearly accessible asymptotic value at ζ, an interior point of $C\left(a, \frac{a+b}{2}\right)$. Let S be an asymptotic path of f ending at ζ so that f maps S one-to-one onto a line segment $f(S)$. Clearly, the arc S must intersect infinitely many of the arcs T_n. Hence, the line segment $f(S)$ must be a segment of the real line K. On the other hand, S must intersect infinitely many of the arcs R_k^*, and hence, $f(S)$ must intersect the lines J_k. This contradiction concludes the proof of Theorem 1.

EXAMPLE. *Theorem 1 is false for meromorphic functions.* Let K denote the real line (in the w-plane). We exhibit a meromorphic function f without asymptotic values such that the set $\tilde{f}^{-1}(F_K)$ ends at points of C. This example is due to Lehto and Virtanen [5, p. 58], and it was used by Barth [1, Example 1] to show that the meromorphic analogue of the statement $\mathscr{A} = \mathscr{L}$ is false. Let g be a modular function omitting the values 0, 1, and i (that is, let g be a one-to-one conformal map of D onto the universal covering surface of the extended complex plane with the points 0, 1, and i removed). Then g is a normal function [5, p. 53] having radial limits at E, a countable dense subset of C. By a theorem of A. J. Lohwater and G. Piranian [6, Theorem 6], there exists a bounded holomorphic function in D having a radial limit at each point of C - E and at no point of E. Let h denote this function, and let r denote a positive number that is an upper bound for $|h|$. The function $f = 3rg + h$ has no radial limits, and it is a normal function, since h is bounded and g is normal [5, p. 53]. Hence, by a theorem of O. Lehto and K. I. Virtanen [5, Theorem 2], f has no asymptotic values.

We now show that the set $\tilde{f}^{-1}(F_K)$ ends at points of C. If $\tilde{f}^{-1}(F_K)$ did not end at points of C, then there would exist a sequence $\{T_n\}$ of arcs in D such that $f(T_n) \subset K$ and $\{T_n\}$ tends to an arc C(a, b) on C. Then we could choose a point $\zeta \in C(a, b)$ and a sequence $\{z_n\}$ of points on the radius of D ending at ζ such that

(i) g has radial limit i at ζ,

(ii) $z_n \in T_n$,

(iii) $z_n \to \zeta$.

On the one hand, $\Im(f(z_n)) = 0$ for each n, since $\Im(f(z)) = 0$ for $z \in T_n$; on the other hand, $\Im(f(z_n)) = 3r\Im(g(z_n)) + \Im(h(z_n)) \geq 2r - r > 0$ for sufficiently large values n. Thus the set $\tilde{f}^{-1}(F_K)$ ends at points of C.

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