PAIRS OF ADDITIVE EQUATIONS

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1. INTRODUCTION

Let

$$F = \sum_{i=1}^{n} a_i x_i^3 \quad \text{and} \quad G = \sum_{i=1}^{n} b_i x_i^3 , \tag{1}$$

where the coefficients are integers. We consider the equations

$$F = G = 0 . \tag{2}$$

H. Davenport and D. J. Lewis [2] proved that if $n \geq 16$, there exists a nontrivial solution of (2) in every p-adic field, and that if $n \geq 18$ there exists a nontrivial solution of (2) in rational integers. By applying an appropriate form of Hua's Lemma, we simplify the analytic part of the argument (particularly the treatment of the minor intervals), and we obtain the following stronger result.

THEOREM 1. *If* $n \geq 17$, *then the equations* (2) *have a nontrivial solution in rational integers.*

More generally, we can consider pairs of additive equations

$$f = \sum_{i=1}^{n} a_i x_i^k = 0 , \quad g = \sum_{i=1}^{n} b_i x_i^k = 0 , \tag{3}$$

where $k \geq 3$ and the coefficients are integers.

THEOREM 2. *If*

(i) *the equations* (3) *have a nonsingular solution in every* p-*adic field and in the real field, if*

(ii) $n > 2^{k+1}$,

and if, in case the degree k *of the equations* (3) *is even,*

(iii) *every member of the pencil* $\{\lambda f + \mu g\}$ $[(\lambda, \mu) \neq (0, 0)]$ *contains at least* $2^k + 1$ *variables with nonzero coefficients,*

then the equations (3) *have a nontrivial solution in rational integers.*

Theorem 2 will not be proved; it can be proved in precisely the same way as Theorem 1. From the p-adic results of Davenport and Lewis [3], we shall deduce the following result, which is new for $k < 12$ but inferior to results of Davenport and Lewis [4] for large k.

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THEOREM 3. *If* k *is odd,* $k \geq 5$, *and* $n > 2^{k+1}$, *then the equations* (3) *have a nontrivial solution in rational integers.*

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2. DEDUCTION OF THEOREM 3 FROM THEOREM 2

For odd k, the equations (3) have a nonsingular real solution. We may suppose that every member of the pencil $\{\lambda f + \mu g\}$ $[(\lambda, \mu) \neq (0, 0)]$ contains at least $2^k + 1$ variables with nonzero coefficients; for otherwise, we could transform the equations (3) into

$$f = \sum_{i=1}^{n} a_i x_i^k = 0, \qquad g = \sum_{i=v+1}^{n} b_i x_i^k = 0,$$

where $v = 2^k + 1$. We take $x_i = 0$ for $i = v + 1, \cdots, n$ and solve the equation

$$a_1 x_1^k + \cdots + a_v x_v^k = 0$$

in integers $x_1, \cdots, x_v$, not all zero, to obtain a nontrivial solution of (3).

Since $2^k \geq k^2$ for $k \geq 4$, Theorem 3 now follows from Theorem 2 and the following two p-adic results of Davenport and Lewis [3].

If k *is odd and* $n \geq 2k^2 + 1$, *then, for each prime* p, *the equations* (3) *have a nontrivial* p-*adic solution.*

If every form $\lambda f + \mu g$ $[(\lambda, \mu) \neq (0, 0)]$ *in the pencil of* f *and* g *has at least* $k^2 + 1$ *variables with nonzero coefficients and the equations* (3) *have a nontrivial* p-*adic solution, then they have a nonsingular* p-*adic solution.*

3. PRELIMINARIES TO THE PROOF OF THEOREM 1

In proving Theorem 1, we may suppose that $n = 17$, since the remaining variables can be taken as zero. We may also suppose that every member of the pencil $\{\lambda f + \mu g\}$ $[(\lambda, \mu) \neq (0, 0)]$ contains at least 9 variables explicitly, so that no ratio occurs more than 8 times among the a_i/b_i in (1).

Since the ratios $a_1/b_1, \cdots, a_{17}/b_{17}$ are not all equal, there exists a real solution of the pair of equations

$$a_1 \chi_1 + \cdots + a_{17} \chi_{17} = 0,$$

$$b_1 \chi_1 + \cdots + b_{17} \chi_{17} = 0,$$

where each χ_i is different from zero. We may also suppose that each of the χ_i is positive. Multiplying the solution by a suitable factor, we can choose an integer C and a real solution such that

$$1 < \chi_i < C^3 \qquad \text{for } i = 1, \cdots, 17 .$$

For $i = 1, \cdots, 17$, we take

$$\gamma_i = a_i \alpha + b_i \beta \tag{4}$$

and

$$T(\gamma_i) = \sum_{x=P}^{CP} e(\gamma_i x^3), \tag{5}$$

where P is a large integer. We take B to be the box defined by the inequalities

$$1 \leq x_i \leq C \quad (i = 1, \cdots, 17).$$

Then the number of simultaneous integer solutions of (2) in PB is given by the expression

$$N(P) = \int_0^1 \int_0^1 \prod_{i=1}^{17} T(\gamma_i)\, d\alpha\, d\beta. \tag{6}$$

We take the major interval $\mathfrak{M}(A, B, R)$ to consist of the (α, β) with rational approximations

$$|\alpha - A/R| < P^{\delta-3}, \quad |\beta - B/R| < P^{\delta-3},$$

where $(A, B, R) = 1$; $1 \leq A, B \leq R$; and $1 \leq R \leq P^{\delta}$. We denote the union of the major intervals by $\mathfrak{M}$. The minor intervals $\mathfrak{m}$ consist of the remainder of the square $0 < \alpha < 1$, $0 < \beta < 1$. Here δ is a small positive constant independent of P. Where we use Vinogradov's $\ll$-notation, the implied constants may depend on F and G as well as on ε.

The following lemma is the corollary to Theorem 1 of Davenport and Lewis [2].

LEMMA 1. *If* $n \geq 16$ *and every member of the pencil*

$$\{\lambda f + \mu g\} \quad [(\lambda, \mu) \neq (0, 0)]$$

contains at least 7 *variables explicitly, then* F *and* G *have a nonsingular simultaneous zero in every* p*-adic field.*

4. THE MINOR INTERVALS

LEMMA 2. *If* $a_i b_j - a_j b_i \neq 0$, *then*

$$\int_0^1 \int_0^1 |T(\gamma_i) T(\gamma_j)|^8 \, d\alpha\, d\beta \ll P^{10+\varepsilon}. \tag{7}$$

Proof. For $0 < \alpha < 1$ and $0 < \beta < 1$, we have the estimates

$$\max(|\gamma_i|, |\gamma_j|) \ll 1$$

and

$$\Delta = |a_i b_j - a_j b_i| = |\partial(\gamma_i, \gamma_j)/\partial(\alpha, \beta)| \ll 1 .$$

Since $\Delta \neq 0$, we can change the variables of integration from α and β to γ_i and γ_j. Using the periodicity of the integrand and applying Hua's Lemma (see [1, Lemma 2], for example), we obtain the estimates

$$\int_0^1 \int_0^1 |T(\gamma_i)\, T(\gamma_j)|^8 \, d\alpha\, d\beta \ll \int_0^1 \int_0^1 |T(\gamma_i)\, T(\gamma_j)|^8 \, d\gamma_i\, d\gamma_j$$

$$\ll \sum_{h=i,j} \left\{ \int_0^1 |T(\gamma_h)|^8 \, d\gamma_h \right\}^2 \ll P^{10+2\varepsilon} .$$

LEMMA 3. *For each* $\varepsilon > 0$,

$$\int_0^1 \int_0^1 \prod_{i=1}^{17}{}' |T(\gamma_i)| \, d\alpha\, d\beta \ll P^{10+\varepsilon} , \tag{8}$$

where ' *indicates the omission of one factor from the product.*

Proof. Since we may assume that at most 8 ratios a_i/b_i are equal, the forms γ_i can be arranged into 8 pairs γ_k, γ_l such that $a_k b_l - a_l b_k \neq 0$. Then, by (7),

$$\int_0^1 \int_0^1 \prod_{i=1}^{17}{}' |T(\gamma_i)| \, d\alpha\, d\beta \ll \sum_{k,l} \int_0^1 \int_0^1 |T(\gamma_k)\, T(\gamma_l)|^8 \, d\alpha\, d\beta \ll P^{10+\varepsilon} .$$

For estimating the contribution of the minor intervals, it is now sufficient to show that on $\mathfrak{m}$ at least one of the $T(\gamma_i)$ is small compared with P.

LEMMA 4. *Suppose that* $(\alpha, \beta) \in \mathfrak{m}$ *and* $a_i b_j - a_j b_i \neq 0$. *Then*

$$\min(|T(\gamma_i)|, |T(\gamma_j)|) < P^{1-\delta/7} . \tag{9}$$

Proof. Suppose the result is false. Let

$$|T(\gamma_i)| = P^{1-\sigma} , \quad |T(\gamma_j)| = P^{1-\tau} ,$$

where $\max(\sigma, \tau) \leq \delta/7$. Then, by Lemma 29 of Davenport and Lewis [2], there exist rational approximations A_i/Q_i and A_j/Q_j to γ_i and γ_j, respectively, such that

$$1 \leq Q_i \ll P^{3\sigma} , \quad |\gamma_i - A_i/Q_i| \ll (Q_i^{1/3} P^{3-\sigma})^{-1} ,$$

$$1 \leq Q_j \ll P^{3\tau} , \quad |\gamma_j - A_j/Q_j| \ll (Q_j^{1/3} P^{3-\tau})^{-1} .$$

Now $(a_i b_j - a_j b_i)\alpha = b_j \gamma_i - b_i \gamma_j$, and a similar equation holds for β. Thus α and β have rational approximations A/R and B/R such that (A, B, R) = 1 and $R \mid (a_i b_j - a_j b_i) Q_i Q_j$. Thus

$$R \ll P^{3(\sigma+\tau)} < P^{\delta} .$$

Also,

$$|\alpha - A/R| \ll |\gamma_i - A_i/Q_i| + |\gamma_j - A_j/Q_j| \ll P^{\sigma+\tau-3},$$

and a similar relation holds for $|\beta - B/R|$. Thus, if $\sigma \leq \delta/7$ and $\tau \leq \delta/7$, then $(\alpha, \beta) \in \mathfrak{M}(A, B, R)$; this gives a contradiction.

LEMMA 5. *The contribution of the minor intervals to the integral* (6) *is* $\ll P^{11+\varepsilon-\delta/7}$.

Proof. Without loss of generality, we may suppose that $a_1 b_2 - a_2 b_1 \neq 0$. Then, by (8) and (9), the contribution of the minor intervals is

$$\ll \max_{\mathfrak{m}} \min(|T(\gamma_1)|, |T(\gamma_2)|) \int_0^1 \int_0^1 \prod_{i=1}^{17}{}' |T(\gamma_i)| \, d\alpha \, d\beta \ll P^{11+\varepsilon-\delta/7}. \tag{10}$$

5. THE MAJOR INTERVALS

We shall give no detailed proofs here, because the treatment of the major intervals is essentially the same as in [2]. By (6) and (10), we see that

$$N(P) = \iint_{\mathfrak{M}} \prod_{i=1}^{17} T(\gamma_i) \, d\alpha \, d\beta + O(P^{11+\varepsilon-\delta/7}). \tag{11}$$

For $(\alpha, \beta) \in \mathfrak{M}(A, B, R)$, we put

$$d_i = \text{g.c.d.}(Aa_i + Bb_i, R), \qquad R = R_i d_i,$$

and

$$\phi = \alpha - A/R, \qquad \psi = \beta - B/R, \qquad \beta_i = a_i \phi + b_i \psi.$$

We choose C_i so that $(C_i, R_i) = 1$ and

$$C_i/R_i = (Aa_i + Bb_i)/R. \tag{12}$$

The following four lemmas are essentially Lemmas 33, 35, 36, and 39 of Davenport and Lewis [2].

LEMMA 6. *If* $(\alpha, \beta) \in \mathfrak{M}(A, B, R)$, *then for* $i = 1, \cdots, 17$,

$$|T(\gamma_i)| \ll R_i^{-1/3} \min(P, P^{-2} |\beta_i|^{-1}).$$

LEMMA 7. $\sum_{A,B} (R_1 \cdots R_{17})^{-1/3} \ll R^{\varepsilon-2}$, *where the summation is over* $1 \leq A \leq R$, $1 \leq B \leq R$, $(A, B, R) = 1$.

LEMMA 8. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{i=1}^{17} \min(P, P^{-2} |\beta_i|^{-1}) \, d\phi \, d\psi \ll P^{11}$ *and*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{i=1}^{17}{}' \min(P, P^{-2} |\beta_i|^{-1}) \, d\phi \, d\psi \ll P^{10},$$

where ' indicates the omission of one factor from the product.

We take $S(a, q) = \sum_{x=1}^{q} e_q(ax^3)$ and

$$I(\beta_i) = \int_P^{CP} e(\beta_i \xi^3)\, d\xi = \frac{1}{3} \int_{P^3}^{(CP)^3} \eta^{-2/3} e(\beta_i \eta)\, d\eta .$$

LEMMA 9. *For* $(\alpha, \beta) \in \mathfrak{M}(A, B, R)$,

$$T(\gamma_i) = R_i^{-1} S(C_i, R_i) I(\beta_i) + O(R_i^{2/3+\varepsilon}) .$$

LEMMA 10. *The contribution of the major intervals to the integral* (6) *is*

$$G(P^\delta) J(P) + O(P^{21/2}) , \tag{13}$$

where

$$G(P^\delta) = \sum_{R \leq P^\delta} \sum_{A,B} \prod_{i=1}^{17} R_i^{-1} S(C_i, R_i) \quad \textit{and} \quad J(P) = \int\int \prod_{i=1}^{17} I(\beta_i)\, d\phi\, d\psi ,$$

the integration being over $\max(|\phi|, |\psi|) < P^{\delta-3}$.

Lemma 10 follows from Lemmas 40 and 41 of Davenport and Lewis [2]. The next can be proved in the same way as Lemma 42 in [2].

LEMMA 11. *As* $P \to \infty$,

$$J(P) \sim KP^{11} , \tag{14}$$

where K *is a positive constant depending on* C.

6. COMPLETION OF THE PROOF OF THEOREM 1

If any ratio occurs more than eight times among the a_i/b_i, Theorem 1 follows from the remarks in Section 2. Otherwise, by (11), (13), and (14), we have the estimate

$$N(P) = KP^{11} G(P^\delta) + o(P^{11}) \quad \text{as } P \to \infty .$$

Continuing the series for $G(P^\delta)$ to infinity, we obtain the formula

$$G = \sum_{R=1}^{\infty} \sum_{A,B} \prod_{i=1}^{17} R_i^{-1} S(C_i, R_i) .$$

Thus

$$G \ll \sum_{R=1}^{\infty} \sum_{A,B} \prod_{i=1}^{17} R_i^{-1/3} \ll \sum_{R=1}^{\infty} R^{\varepsilon-2} .$$

Hence the series is absolutely convergent and $|G - G(P^\delta)| < P^{-\delta/2}$. Now, by (12),

$$R_i^{-1} S(C_i, R_i) = R^{-1} S(Aa_i + Bb_i, R),$$

so that

$$G = \sum_{R=1}^{\infty} \sum_{A,B} R^{-17} \prod_{i=1}^{17} S(Aa_i + Bb_i, R).$$

Writing

$$\chi_p = 1 + \sum_{v=1}^{\infty} \sum_{\substack{A=1 \\ (A,B,p^v)=1}}^{p^v} \sum_{B=1}^{p^v} (p^v)^{-17} \prod_{i=1}^{17} S(Aa_i + Bb_i, p^v),$$

we have the relation $G = \prod_p \chi_p$.

From Lemma 7 it follows that $|\chi_p - 1| \ll p^{\varepsilon - 2}$ for large p. Hence there exists p_0 such that $\prod_{p \geq p_0} \chi_p \geq 1/2$. For each $p < p_0$, there exists a nonsingular p-adic solution of (2), by Lemma 1. Hence $\chi_p > 0$ for such p. Thus $G > 0$, and therefore

$$N(P) = KGP^{11} + o(P^{11}) \quad \text{as } P \to \infty,$$

and $KG > 0$. Thus $N(P) > 0$ for large P, and the proof of Theorem 1 is complete.

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