TWO-SLIT MAPPINGS AND THE MARX CONJECTURE

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1. INTRODUCTION

Let S^* denote the class of functions $f(z) = z + a_2 z^2 + \cdots$ that map the unit disk $|z| < 1$ conformally onto a domain starlike with respect to the origin. Let $k(z) = z(1 - z)^{-2}$ denote the Koebe function. In 1932, A. Marx [5] proved that for each $f \in S^*$, the function $f(z)/z$ is subordinate to $k(z)/z$, and he conjectured that $f'(z)$ is subordinate to $k'(z)$. In other words, the Marx conjecture asserts that if

$$K_r = \{k'(z): |z| \leq r\} \qquad (0 \leq r < 1)$$

and

$$M_r = \{f'(r): f \in S^*\},$$

then $M_r = K_r$ for each r $(0 \leq r < 1)$.

Marx proved this conjecture for all $r \leq \sin \pi/8 = 0.382 \cdots$. R. M. Robinson [7], [8] improved the constant to $(5 - \sqrt{17})/2 = 0.438 \cdots$, and later to 0.6. P. L. Duren [1] made a further improvement to $0.736 \cdots$. R. McLaughlin [6] obtained the same constant with a different method. However, J. A. Hummel [2] has recently constructed a counterexample, which shows that the Marx conjecture is false for all sufficiently large r. Hummel's example is a mapping

$$f(z) = \frac{z}{(1 - ze^{is})^b (1 - ze^{it})^{2-b}} \qquad (0 \leq b \leq 2;\ 0 \leq s, t \leq 2\pi) \tag{1}$$

of the disk onto the plane slit along two rays. The construction is based on a continuity argument that gives a counterexample for sufficiently small values of b. On the other hand, Hummel shows that, for each $r < 1$, every point on the boundary of the Marx region M_r corresponds to a two-slit mapping of the form (1). Thus the Marx problem is actually equivalent to an analogous question concerning two-slit mappings.

We shall prove that for two-slit mappings with equal weights b and $2 - b$, the Marx conjecture is true. This result is stated more precisely below. Recall that if F and G are analytic functions in the unit disk, then F is said to be *subordinate* to G if $F(z) = G(\omega(z))$ for some analytic function ω with $|\omega(z)| \leq |z|$.

THEOREM 1. *If* f *is a function of the form* (1) *with* $b = 1$, *then* f' *is subordinate to* k'.

The proof depends on a study of the valence of the functions k' and f'. In particular, we find that $\sqrt{k'}$ is univalent. In Section 4, we show that $\log k'$ is starlike.

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2. DERIVATIVE OF THE KOEBE FUNCTION

In preparation for the proof of Theorem 1, we study first the range of

$$k'(z) = \frac{1+z}{(1-z)^3} \qquad (|z| < 1).$$

For each complex number α, $k'(z) = \alpha$ if and only if

$$\alpha z^3 - 3\alpha z^2 + (3\alpha + 1)z + (1 - \alpha) = 0. \tag{2}$$

Hence the problem is to determine, for each α, how many roots equation (2) has in the unit disk. For this, we shall use a theorem of M. Marden [4, p. 197].

For each polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, define

$$p^*(z) = z^n \bar{p}(1/z) = \bar{a}_n + \bar{a}_{n-1} z + \cdots + \bar{a}_0 z^n.$$

Starting with a polynomial $p(z) = p_0(z)$, construct the polynomials

$$p_j(z) = a_0^{(j)} + a_1^{(j)} z + \cdots + a_{n-j}^{(j)} z^{n-j} \qquad (j = 1, \cdots, n)$$

by the relation

$$p_{j+1}(z) = \bar{a}_0^{(j)} p_j(z) - a_{n-j}^{(j)} p_j^*(z).$$

Let $\delta_j = a_0^{(j)}$, and let $P_k = \delta_1 \delta_2 \cdots \delta_k$ $(k = 1, \cdots, n)$. Marden's theorem asserts that if q of the products P_k are negative and the remaining $n - q$ are positive, then $p(z)$ has q zeros in $|z| < 1$ and $n - q$ zeros in $|z| > 1$.

Setting $\alpha = u + iv$, one computes for the polynomial (2) the values

$$\delta_1 = 1 - 2u, \qquad \delta_2 = 1 - 4u - 12u^2 - 16v^2, \qquad \delta_3 = 16(2u - 1)(u + 4u^2 + 4u^3 + 8v^2).$$

Clearly, δ_1 is positive in the half-plane $u < 1/2$, and δ_2 is positive inside the ellipse

$$9(u + 1/6)^2 + 12v^2 = 1. \tag{3}$$

The expression δ_3 is positive in the half-plane $u > 1/2$ and in the two regions G_0 and G_1 (see Figure 1) containing an interval of the negative real axis and bounded by the curve $8v^2 = -u(2u + 1)^2$.

The region G_0 lies inside the ellipse (3). Applying Marden's theorem, one obtains the following result.

THEOREM 2. *In the unit disk* $|z| < 1$, *the function* $k'(z)$ *omits all values in the set*

$$G_0 = \{u + iv: -1/2 \le u \le 0,\ v^2 \le -u(2u + 1)^2/8\},$$

it assumes all values in the set

$$G_1 = \{u + iv: u < -1/2,\ v^2 < -u(2u + 1)^2/8\}$$

exactly twice, and it assumes all other values exactly once.

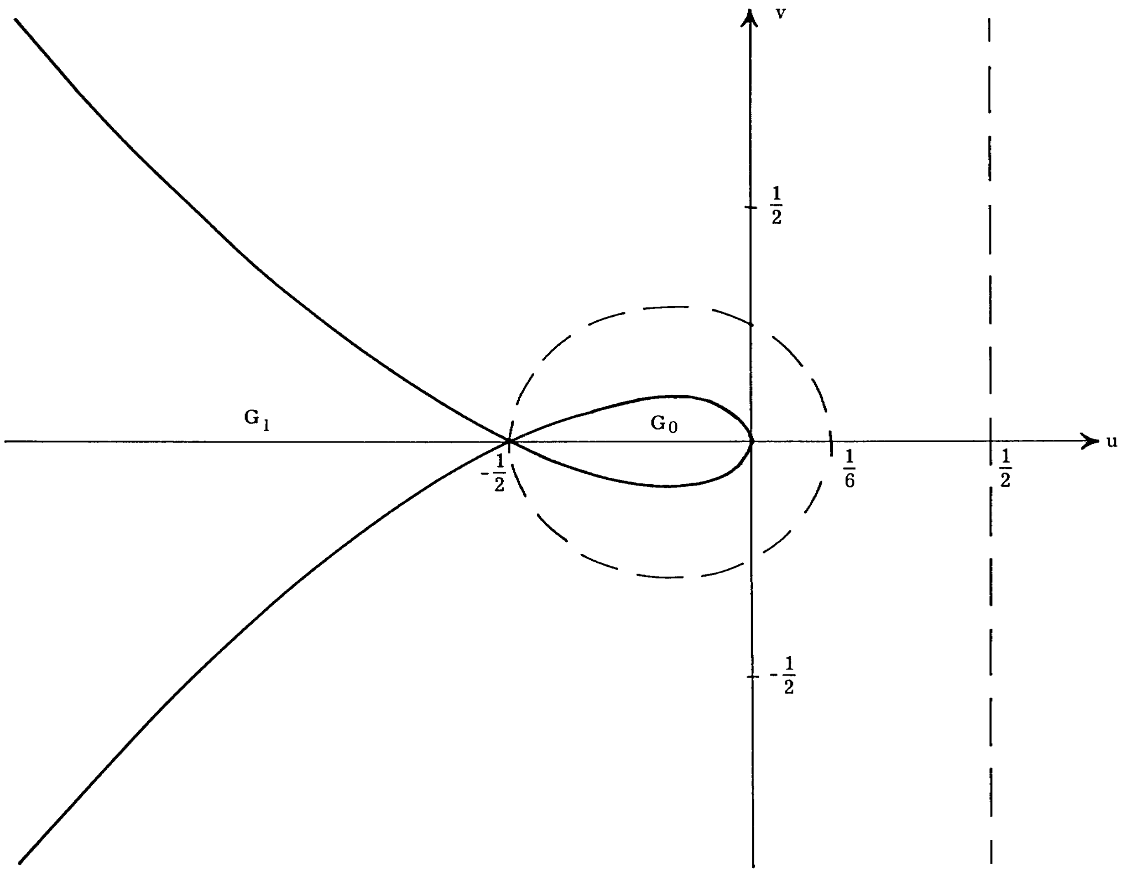

Figure 1. The curve $8v^2 = -u(2u + 1)^2$.

COROLLARY. *The function* $\sqrt{k'(z)}$ *is analytic and univalent in* $|z| < 1$.

3. DERIVATIVE OF TWO-SLIT MAPPINGS

We wish to compare the range of $k'(z)$ with that of $f'(z)$, where

$$f(z) = z(1 - ze^{is})^{-1}(1 - ze^{it})^{-1}.$$

We shall prove Theorem 1 by showing that the range of f' is contained in the range of k', where both ranges are regarded as subsets of a two-sheeted Riemann surface. This will imply that the range of the *univalent* function $\sqrt{k'}$ contains the range of $\sqrt{f'}$. An appeal to Schwarz' lemma then shows that $\sqrt{f'}$ is subordinate to $\sqrt{k'}$, and this proves Theorem 1.

In calculating the range of f', we may assume without loss of generality that $s = 0$ and $0 < t \leq \pi$, so that

$$f'(z) = \frac{1 - \beta z^2}{(1 - z)^2(1 - \beta z)^2} \qquad (\beta = e^{it}). \tag{4}$$

If $t = \pi$, then $f'(z) = (1 + z^2)(1 - z^2)^{-2}$. But it is easy to see (by Marden's theorem, for instance) that the function

$$w = u + iv = \frac{1 + \zeta}{(1 - \zeta)^2}$$

maps the disk $|\zeta| < 1$ conformally onto the parabolic region $u + 2v^2 > 0$, which excludes the region G_0 (see Figure 1). Hence f' is subordinate to k' if $t = \pi$.

Assuming now that $0 < t < \pi$, we use the identity

$$1 - e^{i\theta} = \sqrt{2}\sqrt{1 - \cos\theta}\, e^{i(\theta-\pi)/2} \qquad (0 \leq \theta \leq 2\pi)$$

to express $f'(e^{i\theta})$ in polar form, and we obtain the relation

$$f'(e^{i\theta}) = \frac{\sqrt{2}}{4} \frac{(1 - \cos(2\theta + t))^{1/2}}{(1 - \cos\theta)(1 - \cos(\theta + t))} e^{i\psi(\theta)}, \tag{5}$$

where

$$\psi(\theta) = \begin{cases} -\theta - (t+\pi)/2 & \text{if (i) } 0 < \theta < \pi - t/2, \\ -\theta - (t-\pi)/2 & \text{if (ii) } \pi - t/2 < \theta < 2\pi - t \\ & \text{or (iii) } 2\pi - t < \theta < 2\pi - t/2, \\ -\theta - (t+\pi)/2 & \text{if (iv) } 2\pi - t/2 < \theta < 2\pi. \end{cases} \tag{6}$$

The function $f'(z)$ has zeros at $z = \pm e^{-it/2}$, and poles at $z = 1$ and $z = e^{-it}$. The polar representation (5) shows that f' maps the arcs (i), (ii), (iii), (iv) of the unit circle, as defined in (6) and illustrated in Figure 2, onto curves of the type shown in Figure 3. These curves bound regions R_0, R_1, R_2, R_3, and R_4, as indicated in in Figure 3. A local inspection of f' near the poles 1 and e^{-it} reveals that f' carries small circular arcs centered at these points (as shown in Figure 2) onto curves of the types indicated by dots in Figure 3.

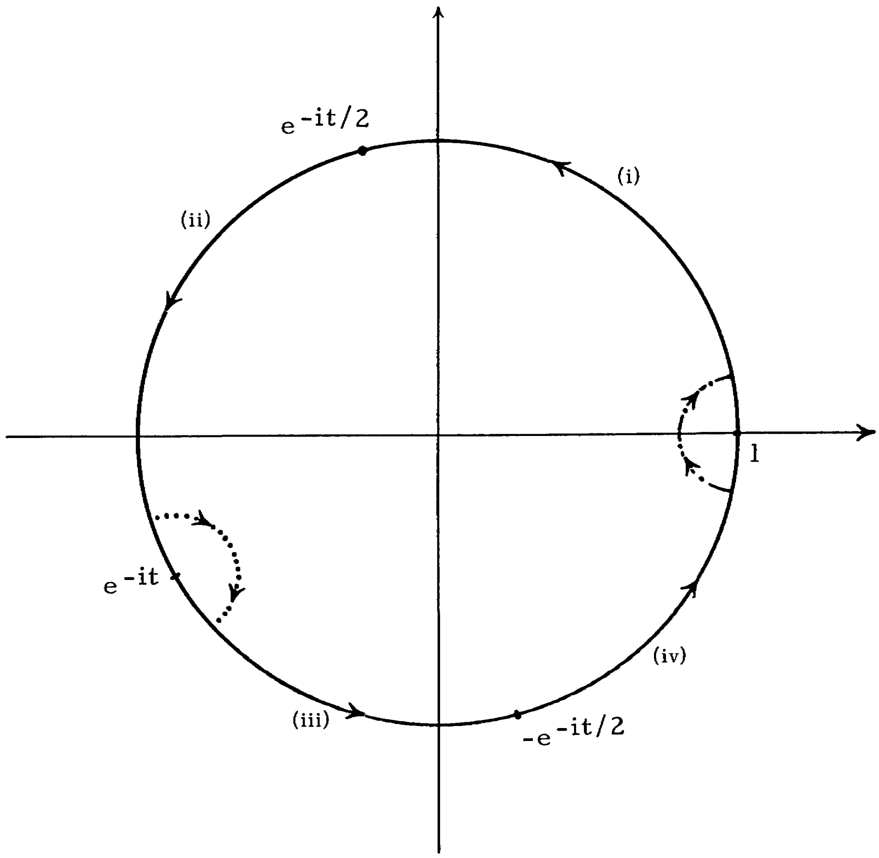

Figure 2. Subarcs of the unit circle.

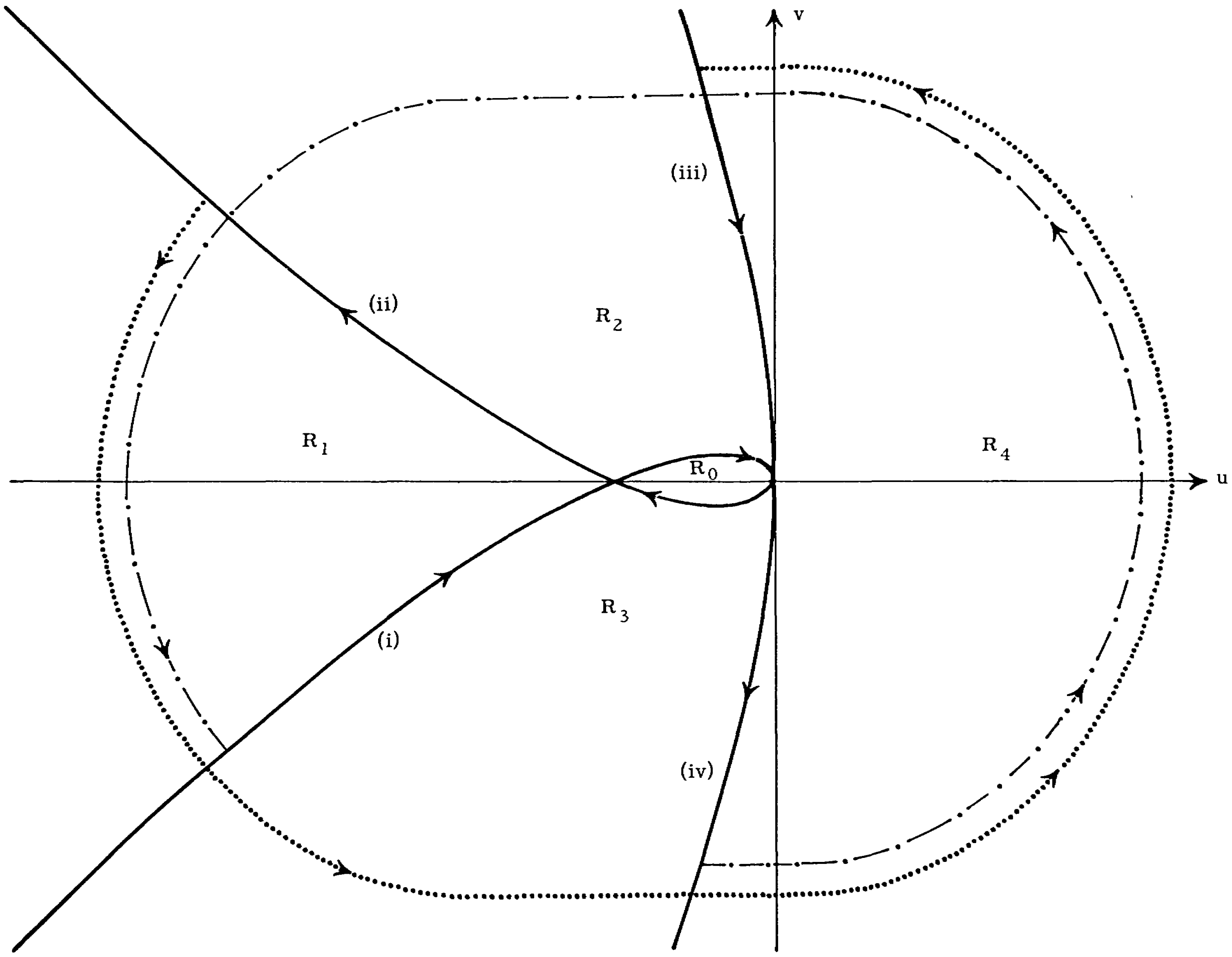

Figure 3. The range of f'.

A study of Figure 3 reveals that for sufficiently small circular arcs circumventing the poles, the image of the modified unit circle winds about each point in R_0, R_1, R_2, R_3, R_4 exactly 0, 2, 1, 1, 2 times, respectively. Thus, by the argument principle, f' maps the unit disk onto a domain that omits R_0, covers R_2 and R_3 once, and covers R_1 and R_4 twice.

Therefore, in order to show that the range of $\sqrt{k'}$ contains that of $\sqrt{f'}$, it will be enough to show that $G_0 \subset R_0$ and $R_1 \subset G_1$. We can establish these inclusions by showing that each ray from the origin meets the boundary of G_0 before it meets that of R_0 and meets the boundary of G_1 before it meets that of R_1. For this purpose, we compare the polar representation of $f'(e^{i\theta})$, as given by (5) and (6), with that of $k'(e^{i\theta})$:

$$k'(e^{i\theta}) = \frac{\sqrt{2}}{4} \frac{(1 - \cos 2\theta)^{1/2}}{(1 - \cos\theta)^2} e^{i\phi(\theta)}, \tag{7}$$

where

$$\phi(\theta) = \begin{cases} -\theta - \frac{\pi}{2} & \text{if } 0 < \theta < \pi, \\ -\theta + \frac{\pi}{2} & \text{if } \pi < \theta < 2\pi. \end{cases} \tag{8}$$

Since the curves $f'(e^{i\theta})$ and $k'(e^{i\theta})$ are both symmetric with respect to the real axis, we may confine attention to the branch of (7) for which $0 < \theta < \pi$, and to the corresponding branch of (5) for which $0 < \theta < \pi - t/2$. For each θ in the interval $(0, \pi)$ for which the ray from the origin with argument $\phi(\theta)$ intersects the arc (i) of the boundary curve of f', let θ^* be determined so that $\phi(\theta) = \psi(\theta^*)$ and $0 < \theta^* < \pi - t/2$; that is, $\theta^* = \theta - t/2$ and $t/2 < \theta < \pi$. Now we have only to show that

$$|k'(e^{i\theta})| \leq |f'(e^{i\theta^*})| \qquad (t/2 < \theta < \pi). \tag{9}$$

But (9) is equivalent to the elementary inequality

$$[1 - \cos(\theta - \alpha)][1 - \cos(\theta + \alpha)] \leq [1 - \cos\theta]^2,$$

where $\alpha < \theta < \pi$ and $0 < \alpha = t/2 < \pi/2$. This completes the proof of Theorem 1.

4. STARLIKENESS OF $\log k'(z)$

Although $k'(z)$ is not univalent, it is interesting to observe that $\log k'(z)$ is univalent and maps the unit disk onto a starlike domain. In other words, $\log \sqrt[4]{k'(z)} \in S^*$. The starlikeness is an immediate consequence of the polar representation (7), which gives the relation

$$\log k'(e^{i\theta}) = \log \rho(\theta) + i\phi(\theta),$$

where $\rho(\theta) = |k'(e^{i\theta})|$. The appropriate form of ϕ is now

$$\phi(\theta) = \begin{cases} 3\pi/2 - \theta & \text{if } 0 < \theta < \pi, \\ \pi/2 - \theta & \text{if } \pi < \theta < 2\pi. \end{cases}$$

In the interval $(0, \pi)$, $\log \rho(\theta)$ decreases from $+\infty$ to $-\infty$, while $\phi(\theta)$ decreases from $3\pi/2$ to $\pi/2$. In the interval $(\pi, 2\pi)$, $\log \rho(\theta)$ increases from $-\infty$ to $+\infty$, while $\phi(\theta)$ decreases from $-\pi/2$ to $-3\pi/2$. Hence these two curves bound a strip that is starlike with respect to the origin. The actual domain is shown in Figure 4.

Professor M. S. Robertson has communicated to us the following proof that $w = \log k'(z)$ is close-to-convex (see W. Kaplan [3]), and therefore univalent. Let

$$F(z) = \log k'(z), \qquad G(z) = \frac{1}{2}\log\frac{1+z}{1-z}.$$

Then $w = G(z)$ is a normalized convex mapping, and the function

$$\frac{F'(z)}{G'(z)} = 2(2 + z)$$

has positive real part.

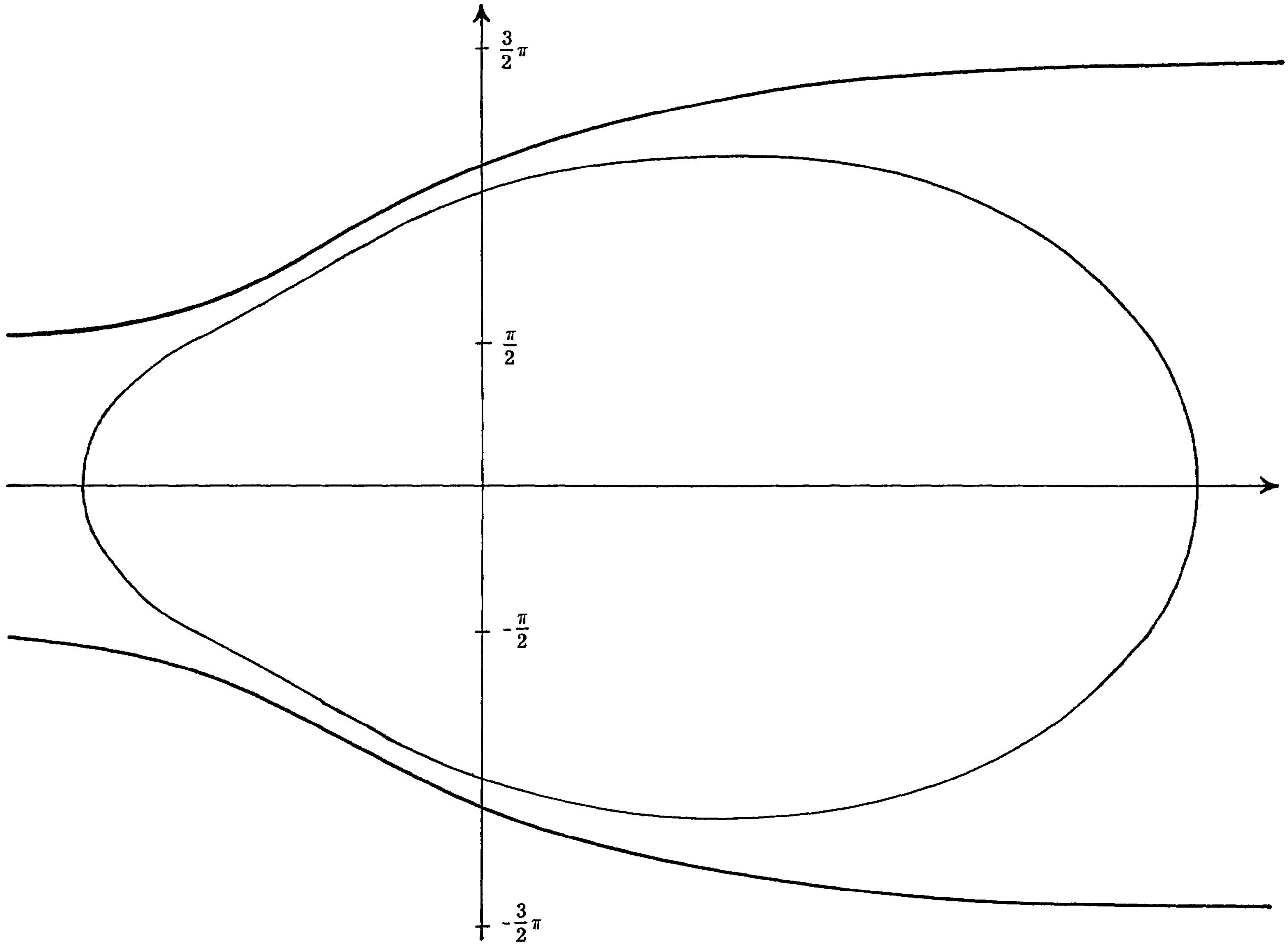

Figure 4. The images of the circles $|z| = 1$ and $|z| = 0.9$ under the mapping $w = \log k'(z)$.

REFERENCES

1. P. L. Duren, *On the Marx conjecture for starlike functions.* Trans. Amer. Math. Soc. 118 (1965), 331-337.

2. J. A. Hummel, *The Marx conjecture for starlike functions.* Michigan Math. J. 19 (1972), 257-266.

3. W. Kaplan, *Close-to-convex schlicht functions.* Michigan Math. J. 1 (1952), 169-185.

4. M. Marden, *Geometry of polynomials.* Second Edition. American Mathematical Society, Providence, R.I., 1966.

5. A. Marx, *Untersuchungen über schlichte Abbildungen.* Math. Ann. 107 (1932), 40-67.

6. R. McLaughlin, *On the Marx conjecture for starlike functions of order* α. Trans. Amer. Math. Soc. 142 (1969), 249-256.

7. R. M. Robinson, *Univalent majorants.* Trans. Amer. Math. Soc. 61 (1947), 1-35.

8. ———, *Extremal problems for star mappings.* Proc. Amer. Math. Soc. 6 (1955), 364-377.

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