LIMITS OF NILPOTENT AND QUASINILPOTENT OPERATORS

James H. Hedlund

The structure of the set of nilpotent operators on an infinite-dimensional Hilbert space is still incompletely described. Many natural questions, suggested by finite-dimensional results, remain to be answered. One such question, raised by P. R. Halmos [2, Question 7], asks for a description of the closure of the nilpotent operators in the uniform operator topology. In this paper, we show that most self-adjoint operators are not limits of nilpotent or quasinilpotent operators, but that many interesting (and not quasinilpotent) weighted shifts are. The results suggest that a simple characterization of the closure of the nilpotent operators may be difficult to discover.

Throughout this paper, H will be a separable complex Hilbert space, usually infinite-dimensional, and an operator will be a bounded linear transformation on H. We shall denote by $\mathcal{N}$ the set of nilpotent operators on H (all operators S with $S^k = 0$ for some k), by $\mathcal{Q}$ the set of quasinilpotent operators (all S with $r(S) = \lim \|S^k\|^{1/k} = 0$), and by $\mathcal{N}^-$ and $\mathcal{Q}^-$ the respective closures in the uniform operator topology. If H is finite-dimensional, then $\mathcal{N}^- = \mathcal{N} = \mathcal{Q} = \mathcal{Q}^-$; in general, $\mathcal{N}$ is properly contained in $\mathcal{Q}$. It is still unknown whether $\mathcal{Q} \subset \mathcal{N}^-$. We follow the notation of [1, p. 37] for the various parts of the spectrum of an operator: Λ will denote the spectrum, Π_0 the point spectrum, Π the approximate point spectrum, and Γ the compression spectrum.

1. SPECTRAL PROPERTIES OF $\mathcal{N}^-$ AND $\mathcal{Q}^-$

Since the quasinilpotent operators are precisely those whose spectrum is the single point $\{0\}$, the problem of characterizing $\mathcal{Q}^-$ is that of describing which operators can be approximated by operators with spectrum $\{0\}$. Clearly, the spectral radius is discontinuous near such operators. It is thus natural to expect that spectral properties will give partial and incomplete information about $\mathcal{Q}^-$. Note that it suffices to investigate operators of norm 1, since $\mathcal{N}$ and $\mathcal{Q}$ are closed under multiplication by scalars.

PROPOSITION 1. *If* T *is bounded below by* ε, *then*

$$d(T, \mathcal{Q}) = \inf\{\|T - S\| : S \in \mathcal{Q}\} \geq \varepsilon .$$

Proof. If S is quasinilpotent, then $0 \in \Pi(S)$. Thus there exists a sequence of vectors $\{x_n\}$ with $\|x_n\| = 1$ and $\|Sx_n\| \to 0$. Hence

$$\|(T - S)x_n\| \geq \|Tx_n\| - \|Sx_n\| \geq \varepsilon - \|Sx_n\| \to \varepsilon ,$$

so that $\|T - S\| \geq \varepsilon$.

COROLLARY 1. *If* T *is invertible, then* $T \notin \mathcal{Q}^-$. *Equivalently, if* $T \in \mathcal{Q}^-$, *then* $0 \in \Lambda(T)$.

Received January 19, 1972.

Michigan Math. J. 19 (1972).

COROLLARY 2. *If* T *is an isometry, then* $d(T, \mathcal{Q}) = 1$.

PROPOSITION 2. *If* $T \in \mathcal{Q}^-$, *then* $\Lambda(T) = \Pi(T)$.

Proof. If $0 \neq \lambda \in \Lambda(T) \setminus \Pi(T)$, then, by the semicontinuity of $\Lambda(T) \setminus \Pi(T)$ [3, Theorem 3], $\lambda \in \Lambda(S) \setminus \Pi(S)$ for all S sufficiently near T. Thus S cannot be quasinilpotent, and $T \notin \mathcal{Q}^-$. Since $\Pi(T)$ is closed and $\Pi(T) \supset \operatorname{bdy} \Lambda(T)$, the same conclusion holds for $\lambda = 0$.

The best-known operators that satisfy the conditions of Propositions 1 and 2 are the noninvertible normal operators. It is clear that there are no nontrivial quasinilpotent normal operators, and the results of the remainder of this section suggest that there may be no normal operators in $\mathcal{Q}^-$ either.

PROPOSITION 3. *If* $0 \neq P$ *is a projection, then* $d(P, \mathcal{Q}) \geq 1/4$.

Proof. If $P = I$, then Corollary 2 applies. Otherwise let K be the range of P, and let T be any operator with $\|P - T\| = \varepsilon < 1/4$. We shall show that $T \notin \mathcal{Q}$, by finding a lower bound on $\|T^n x\|$ for $x \in K$. Write $H = K \oplus K^\perp$, and write

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

relative to this decomposition. Thus $\|I - A\| \leq \varepsilon$ and B, C, and D each have norm at most ε. Let $P^\perp = I - P$.

Let $x \in H$ with $\|Px\| > \|P^\perp x\|$. We begin by proving that

$$\|Tx\| > 2^{-1/2}(1 - 2\varepsilon)\|x\|$$

and that $\|PTx\| > \|P^\perp Tx\|$. Indeed, the inequalities

$$\|APx\| \geq \|IPx\| - \|(I - A)Px\| \geq (1 - \varepsilon)\|Px\|$$

imply that

$$\|PTx\| = \|APx + CP^\perp x\| \geq \|APx\| - \|CP^\perp x\| \geq (1 - \varepsilon)\|Px\| - \varepsilon\|P^\perp x\|$$
$$> (1 - 2\varepsilon)\|Px\| .$$

Since $\|x\|^2 = \|Px\|^2 + \|P^\perp x\|^2$ and $\|Px\| > \|P^\perp x\|$, it follows that $\|Px\| > 2^{-1/2}\|x\|$. Consequently, $\|PTx\| > 2^{-1/2}(1 - 2\varepsilon)\|x\|$, and the first observation is proved. For the second,

$$\|P^\perp Tx\| = \|BPx + DP^\perp x\| \leq \|BPx\| + \|DP^\perp x\| \leq \varepsilon\|Px\| + \varepsilon\|P^\perp x\| < 2\varepsilon\|Px\| .$$

Since $\varepsilon < 1/4$, $\|P^\perp Tx\| < 2\varepsilon\|Px\| < (1 - 2\varepsilon)\|Px\| < \|PTx\|$.

Now let $x \in K$ with $\|x\| = 1$. By the last paragraph, $\|Tx\| > 2^{-1/2}(1 - 2\varepsilon)$ and $\|PTx\| > \|P^\perp Tx\|$. It follows by induction that $\|T^n x\| > (2^{-1/2}(1 - 2\varepsilon))^n$, so that $\|T^n\|^{1/n} > 2^{-1/2}(1 - 2\varepsilon) > 0$. Thus $T \notin \mathcal{Q}$.

Let $\mathcal{P}$ denote the set of all projections with the exception of the operator 0. Proposition 3 then asserts that $d(\mathcal{P}, \mathcal{Q}) \geq 1/4$. The determination of the precise value of $d(\mathcal{P}, \mathcal{Q})$ seems difficult, even if H is finite-dimensional. The case of dimension 2 is tractable, though.

PROPOSITION 4. *If* $T^2 = 0$ *and if* $0 \neq P$ *is a projection, then* $\|P - T\| \geq 2^{-1/2}$.

Proof. If $T = 0$, then $\|P - T\| = 1$. Generally let $K = \ker T$ and suppose that $\|P - T\| < 2^{-1/2}$. If $x \in K$ and $y \in K^{\perp}$, then $\|Px\| = \|(P - T)x\| < 2^{-1/2}\|x\|$ and $\|Py\| = \|P^* y\| = \|(P - T)^* y\| < 2^{-1/2}\|y\|$. Let $z \in H$ with $\|z\| = 1$ and $Pz = z$. Write $z = x + y$, where $x \in K$ and $y \in K^{\perp}$. Since $\|x\|^2 + \|y\|^2 = 1$, it follows that $\|x\| + \|y\| \leq 2^{1/2}$. Now

$$1 = \|Pz\| = \|Px + Py\| \leq \|Px\| + \|Py\| < 2^{-1/2}(\|x\| + \|y\|) \leq 1,$$

which is a contradiction. Thus $\|P - T\| \geq 2^{-1/2}$.

COROLLARY 1. *If* $\dim H = 2$, *then* $d(\mathscr{P}, \mathscr{Q}) = 2^{-1/2}$.

Proof. A quasinilpotent operator on a space of dimension 2 is nilpotent of index 2; thus Proposition 4 applies. The opposite inequality follows from the example

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},$$

where $T^2 = 0$ and $\|P - T\| = 2^{-1/2}$.

COROLLARY 2. *If* H *is a Hilbert space, then* $1/4 \leq d(\mathscr{P}, \mathscr{Q}) \leq 2^{-1/2}$.

Since the estimates used in Proposition 3 are rather crude, it seems likely that the upper bound obtained for $d(\mathscr{P}, \mathscr{Q})$ is more accurate than the lower bound.

PROPOSITION 5. *If* $0 \neq P$ *is a projection and* $\|S\| < 1$, *then* $P \oplus S \notin \mathscr{Q}^-$.

Proof. Since $\|S\| < 1$, there exists an integer m such that $\|S^m\| < 1/4$. Then

$$\|(P \oplus S)^m - (P \oplus 0)\| = \|P \oplus S^m - P \oplus 0\| = \|S^m\| < 1/4,$$

and $P \oplus 0$ is a projection. Thus $(P \oplus S)^m \notin \mathscr{Q}^-$, by Proposition 3. If $P \oplus S \in \mathscr{Q}^-$, then there exist quasinilpotent operators T_n with $T_n \to P \oplus S$. But T_n^m is also quasinilpotent and $T_n^m \to (P \oplus S)^m$, a contradiction.

THEOREM 1. *If* S *is self-adjoint with* $\|S\| = 1$, *and if* 1 *is an isolated point of* $\Lambda(S^2)$ $(= \Lambda(S)^2)$, *then* $S \notin \mathscr{Q}^-$.

Proof. By the Spectral Theorem, $S^2 = S_1 \oplus S_2$, where each S_i is self-adjoint, $\Lambda(S_1) = \{1\}$, and $\|S_2\| < 1$. Proposition 5 now implies that $S^2 \notin \mathscr{Q}^-$. As in the proof of Proposition 5, it follows that $S \notin \mathscr{Q}^-$.

Note that an improvement in the estimates of Proposition 3 would yield an improvement in Theorem 1: every positive self-adjoint operator is within 1/2 of a projection.

2. WEIGHTED SHIFTS

S. Kakutani used a weighted shift to give the basic example of an operator in $\mathscr{N}^-$ that is not quasinilpotent. We exploit the relatively transparent structure of weighted shifts to construct a large class of operators in $\mathscr{N}^- \setminus \mathscr{Q}$.

If $\{e_n\}_0^\infty$ is an orthonormal basis for H and if $\{\omega_n\}_0^\infty$ is a bounded sequence of complex numbers, then the equations

$$We_n = \omega_n e_{n+1} \quad (n = 0, 1, 2, \cdots)$$

define a (unilateral) weighted shift W. We observe the usual convention that $0 \leq \omega_n \leq 1$ for all n (since any weighted shift of norm at most 1 is unitarily equivalent to a weighted shift satisfying these conditions [1, Problem 75]). We will allow some weights to equal 0. Proposition 6 summarizes some well-known information.

PROPOSITION 6. *Let* W *be a weighted shift with weights* $\{\omega_n\}_0^\infty$.

i) W *is nilpotent if and only if blocks of consecutive nonzero weights are bounded in length: there exists a positive integer* k *such that each set of indices* $\{n, n+1, \cdots, n+k-1\}$ $(n = 0, 1, 2, \cdots)$ *contains an index* j *for which* $\omega_j = 0$.

ii) W *is quasinilpotent if and only if*

$$r(W) = \lim_k \sup_n \left(\prod_{j=0}^{k-1} \omega_{n+j}\right)^{1/n} = 0.$$

For a proof, see [1, Problem 77].

COROLLARY. *If a weighted shift* W *is quasinilpotent, then blocks of consecutive large weights are bounded in length:*

(*) *for every* $\varepsilon > 0$ *there exists a positive integer* k *such that each set of indices* $\{n, n+1, \cdots, n+k-1\}$ $(n = 0, 1, 2, \cdots)$ *contains an index* j *for which* $\omega_j < \varepsilon$.

PROPOSITION 7. *If a weighted shift* W *satisfies* (*), *then* W *is the limit of a sequence of nilpotent weighted shifts. In particular, if* $W \in \mathcal{Q}$, *then* $W \in \mathcal{N}^-$.

Proof. Given $\varepsilon > 0$ and W satisfying (*), let X be the shift with weights $\{\chi_n\}$ defined by $\chi_n = \omega_n$ if $\omega_n \geq \varepsilon$ and $\chi_n = 0$ if $\omega_n < \varepsilon$. Then $\|W - X\| \leq \varepsilon$, and Proposition 6 and its corollary show that X is nilpotent.

Kakutani's example [1, Problem 87], [4, p. 282] is the shift whose weights are defined as follows: every second ω_n is equal to 1, every second of the remaining ω_n is equal to 1/2, every second of the remaining ω_n is equal to 1/4, and so forth. Thus the shift satisfies (*) and belongs to $\mathcal{N}^-$; a calculation shows that its spectral radius is 1. It is perhaps tempting to hope that the weighted shifts in $\mathcal{N}^-$ are precisely those satisfying (*). Theorem 2 shows that this is decidedly false.

THEOREM 2. *Let* V *be a weighted shift with weights* $\{\nu_n\}_0^\infty$, *where* $\nu_n = 0$ *or* 1 *for all* n. *Define a block of* V *of length* m *to be a set of* m *consecutive indices* $\{n+1, n+2, \cdots, n+m\}$ *such that*

$$\nu_{n+1} = \cdots = \nu_{n+m-1} = 1 \quad \textit{and} \quad \nu_n = \nu_{n+m} = 0.$$

If there exist integers m_i *such that* $m_i \to \infty$ *and such that for each* i *the shift* V *has an infinite number of blocks of length* m_i, *then* $V \in \mathcal{N}^-$.

Proof. Note that the hypotheses on V assure that $\|V\| = 1$ and $r(V) = 1$; thus V is not quasinilpotent.

Let $0 < \varepsilon < 1$ be given. Choose an integer k large enough so that $(1 - \varepsilon)^k < \varepsilon^2$ and so that the shift V has an infinite number of blocks of length k. We shall construct an operator T that is nilpotent of index 2k and satisfies the condition $\|V - T\| = \varepsilon\sqrt{2}$.

Since each ν_n is either 0 or 1, the index set of the nonnegative integers is partitioned uniquely into a set of blocks of varying lengths (where a block corresponds to an unbroken sequence of weights of 1 with initial and terminal weights of 0; the terminal index is considered to belong to the block, but the initial index is not) together with, perhaps, some remaining indices corresponding to weights of 0. (These occur whenever there exist two or more consecutive weights of 0.) Let R be the set of these remaining indices, define $Te_n = Ve_n = 0$ if $n \in R$, and define $Te_n = Ve_n$ if n belongs to a block of length less than k. The indices where T is still undefined all belong to blocks of length at least k. Arrange these blocks into two sequences: let $\{B_i\}_1^\infty$ be an enumeration of all blocks of length k, and let $\{C_j\}_1^\infty$ be an enumeration of all blocks of length greater than k. The hypotheses and the choice of k ensure that both collections are infinite.

Let p(j) be the length of the block C_j and let $[\,\cdot\,]$ denote the greatest-integer function. Associate with C_1 the first $[(p(1) - 1)/k]$ of the blocks B_i, then associate with C_2 the next $[(p(2) - 1)/k]$ of the blocks B_i, and so forth. Thus each C_j is associated with $[(p(j) - 1)/k]$ of the blocks B_i, and each B_i is associated with one block C_j.

Let C be an arbitrary block C_j, p its length, $m = [(p-1)/k]$, and $\{B_1, \cdots, B_m\}$ the blocks B_i associated with C. Let $\{g_1, \cdots, g_p\}$ be the basis vectors to which the indices in C correspond (if C consists of the indices $n+1, \cdots, n+p$, then write $g_s = e_{n+s}$), and for $1 \le i \le m$, let $\{f_1^i, \cdots, f_k^i\}$ be the basis vectors to which B_i corresponds. Let $\delta = (1 - \varepsilon)^k/\varepsilon$, and note that $\delta < \varepsilon$. For $0 \le r \le m - 1$ define

$$\begin{aligned} Tg_{rk+1} &= (1 - \varepsilon)g_{rk+2} + \varepsilon f_2^{r+1}, \\ Tg_{rk+n} &= (1 - \varepsilon)g_{rk+n+1} && \text{if } 2 \le n \le k, \\ Tf_n^{r+1} &= f_{n+1}^{r+1} && \text{if } 1 \le n \le k - 1, \\ Tf_k^{r+1} &= -\delta g_{(r+1)k+1}. \end{aligned}$$

Finally, define $Tg_n = Vg_n = g_{n+1}$ if $mk + 1 \le n < p$ and $Tg_p = Vg_p = 0$. The operator T is now defined on each basis vector of H, and by linearity T is defined on all of H. It remains to verify that $T^{2k} = 0$ and that $\|V - T\| = \varepsilon\sqrt{2}$.

Nilpotency: Clearly, if n belongs to a block of length less than k or if $n \in R$, then $T^k e_n = 0$. On a typical set of blocks C and B_i,

$$T^k g_1 = T^{k-1}((1 - \varepsilon)g_2 + \varepsilon f_2^1) = (1 - \varepsilon)^k g_{k+1} + \varepsilon Tf_k^1 = (1 - \varepsilon)^k g_{k+1} - \varepsilon\delta g_{k+1} = 0.$$

Similarly, $T^k g_{rk+1} = 0$ for $0 \le r \le m - 1$. If $n > (m - 1)k$, then $T^{2k} g_n = 0$, since $T = V$ on all these g_n and since the block C has fewer than 2k remaining terms. If $n < (m - 1)k$, then $n = rk + s$ for $1 < s < k$. Hence

$$T^{k+1-s} g_n = (1 - \varepsilon)^{k+1-s} g_{(r+1)k+1},$$

so that $T^{2k+1-s}g_n = 0$. Finally, $T^{k+1-s}f_s^i = -\delta g_{ik+1}$, so that $T^{2k+1-s}f_s^i = 0$. Hence $T^{2k}e_n = 0$ for all basis vectors e_n, and T is nilpotent.

Norm: Observe that the subspace corresponding to the block C and associated blocks $\{B_1, \cdots, B_m\}$ is reducing for both V and T, two subspaces corresponding to different blocks C are orthogonal, and $T = V$ on the orthogonal complement of the span of all such subspaces. Thus it suffices to consider $V - T$ on a single such subspace. Now

$$(V - T)g_n = \begin{cases} \varepsilon g_{n+1} - \varepsilon f_2^{r+1} & (n = rk + 1,\ r < m), \\ \varepsilon g_{n+1} & (n \not\equiv 1 \ (\mathrm{mod}\ m),\ n < mk + 1), \\ 0 & (n \geq mk + 1); \end{cases}$$

$$(V - T)f_n^i = \begin{cases} 0 & (n \neq k), \\ \delta g_{ik+1} & (n = k). \end{cases}$$

Since the images of these basis vectors are mutually orthogonal, the norm of $V - T$ is given by

$$\|V - T\| = \|\varepsilon g_{n+1} - \varepsilon f_2^{r+1}\| = \varepsilon\sqrt{2}.$$

The proof is complete.

Note that the approximating nilpotent operators T are not weighted shifts relative to the basis $\{e_n\}$ but are essentially weighted shifts of multiplicity 2. Let C and $\{B_1, \cdots, B_m\}$ be a typical set of associated blocks, and suppose that the length p of C is exactly $mk + 1$. Suppose also that B_m is followed by an additional weight of 0, and let f_{k+1}^m be the basis vector corresponding to the final 0. For $1 \leq j \leq p$, let H_j be the 2-dimensional subspace with basis $\{h_j^1, h_j^2\}$ defined by $h_j^1 = g_j$, $h_j^2 = f_n^{i+1}$, where $j = ik + n$, $n \leq k$, and $h_p^2 = f_{k+1}^m$. Let S be the finite shift of multiplicity 2 defined by $Sh_j^i = h_{j+1}^i$ for $j < p$ and $Sh_p^i = 0$. Thus $S(H_j) = H_{j+1}$ for $j < p$ and $S(H_p) = 0$. We may then write $V = SY$ and $T = SX$, where X and Y are 2-dimensional "diagonal" operators (each H_j reduces both X and Y) whose action is described by the matrices of $X_j = X \mid H_j$ and $Y_j = Y \mid H_j$:

$$X_j = \begin{pmatrix} 1-\varepsilon & 0 \\ \varepsilon & 1 \end{pmatrix}, \quad Y_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } j \equiv 1 \ (\mathrm{mod}\ k),$$

$$X_j = \begin{pmatrix} 1-\varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad Y_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } j \not\equiv 0 \text{ or } 1 \ (\mathrm{mod}\ k),$$

$$X_j = \begin{pmatrix} 1-\varepsilon & -\delta \\ 0 & 0 \end{pmatrix}, \quad Y_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if } j \equiv 0 \ (\mathrm{mod}\ k),\ j \neq p,$$

$$X_p = 0, \qquad Y_p = 0.$$

The verification that $T^{2k} = 0$ and that $\|V - T\| = \varepsilon\sqrt{2}$ is now immediate. In general (if $p \not\equiv 1 \pmod{k}$), we can achieve this representation of the approximating operators T only if the blocks $\{B_i\}$ can be extended by adding $p - mk$ basis vectors from the remainder set R (which may not be large enough). Also note that the particular representation of V depends strongly on k, and thus on ε.

The technique of the proof of Theorem 1 applies to a weighted shift with blocks of infinite length, that is, to an operator of the form $V \oplus X$, where X is the direct sum of a number of copies of the usual (unweighted) shift and where V is defined in Theorem 2. Indeed, with obvious modifications, the proof also applies to the more general situation of Theorem 3.

THEOREM 3. *Let* X *be a weighted shift, or a direct sum of at most countably many weighted shifts, with* $\|X\| \leq 1$, *and let* V *satisfy the hypotheses of Theorem* 2. *Then* $V \oplus X \in \mathscr{N}^-$.

Note that the operators V of Theorem 2 are all partial isometries (and so are V^n for $n = 2, 3, \cdots$). Thus there are numerous partial isometries in $\mathscr{N}^-$, in contrast to the absence of isometries (Corollary 2 to Proposition 1).

Note also that if in Theorem 3 we set $X = U$ (the unweighted shift), then it follows that $V \oplus U \in \mathscr{N}^-$ while $U \notin \mathscr{Q}^-$. Thus neither $\mathscr{N}^-$ nor $\mathscr{Q}^-$ is preserved by restriction to a reducing subspace.

Finally, note that the operators V of Theorem 2 have spectral parts $\Lambda(V) = \Pi(V) = \{|\lambda| \leq 1\}$ and $\Pi_0(V) = \Gamma(V) = \{0\}$ [5]. The same is true of any weighted shift with weights 0 and 1 only, which is neither nilpotent nor equal to U. It seems unlikely that all such weighted shifts are in $\mathscr{N}^-$; in fact, we conjecture that the conditions of Theorem 1 are necessary as well as sufficient. Thus the partially isometric weighted shifts are likely candidates for a class of operators of which all have identical spectral parts, some belong to $\mathscr{N}^-$, and some do not.

REFERENCES

1. P. R. Halmos, *A Hilbert space problem book.* Van Nostrand, Princeton, N.J., 1967.

2. ———, *Ten problems in Hilbert space.* Bull. Amer. Math. Soc. 76 (1970), 887-933.

3. P. R. Halmos and G. Lumer, *Square roots of operators.* II. Proc. Amer. Math. Soc. 5 (1954), 589-595.

4. C. E. Rickart, *General theory of Banach algebras.* Van Nostrand, Princeton, N.J., 1960.

5. W. C. Ridge, *Approximate point spectrum of a weighted shift.* Trans. Amer. Math. Soc. 147 (1970), 349-356.

University of Massachusetts
Amherst, Massachusetts 01002