THE SUM OF SOLID SPHERES

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1. INTRODUCTION

A *crumpled cube* or *solid sphere* K is a space homeomorphic to the union of a 2-sphere (topologically embedded in the 3-sphere S^3) and one of its complementary domains. The *interior of* K, denoted by Int K, is the set of points where K is a 3-manifold (without boundary), and the *boundary* of K, denoted by Bd K, is the 2-sphere K - Int K. With each pair of crumpled cubes K_1 and K_2 and each homeomorphism h: Bd $K_1 \to$ Bd K_2 we associate a space, denoted by $K_1 \cup_h K_2$ and called the *sum* of K_1 and K_2 by h. It is obtained from the disjoint union of K_1 and K_2 by identification of each point p in Bd K_1 with its image h(p) in Bd K_2.

The space $K_1 \cup_h K_2$ may not be a 3-manifold; but its multiplication with E^1 always yields a manifold [11], namely $S^3 \times E^1$; this indicates that each space $K_1 \cup_h K_2$ behaves much like S^3. In fact, a result of S. Armentrout [2] says that if $K_1 \cup_h K_2$ is a manifold, then it is S^3. Many of the interesting upper-semicontinuous decompositions of S^3 may be viewed as the sum of two crumpled cubes [15], and conversely, the sum of two crumpled cubes is always the decomposition space of some u.s.c. decomposition of S^3 [18].

In Theorem 3 we characterize the sums of crumpled cubes that are topologically equivalent to S^3. The theorem says that $K_1 \cup_h K_2$ is S^3 if and only if h mismatches two special 0-dimensional F_σ-sets in the boundaries of K_1 and K_2. We present some applications and corollaries to this mismatch theorem in Section 6, and in Section 3 we reduce the sufficiency to the main lemma of Section 4. In Section 5, we reduce the necessity to the 2-sided approximation theorem of [14], and in Section 2 we give some preliminary information.

2. 0-DIMENSIONAL F_σ-SETS IN THE BOUNDARY OF A CRUMPLED CUBE

N. Hosay [17] and L. L. Lininger [18] have shown that each crumpled cube K can be embedded in S^3 so that Cl$(S^3$ - K) is a 3-cell. Consequently, the following definition has meaning in connection with all crumpled cubes.

Definition. A closed set X in the boundary of a crumpled cube K is *tame*, provided every embedding of K in S^3 such that Cl$(S^3$ - K) is a 3-cell carries X into a 2-sphere in S^3 that is tame in the usual sense.

Because of its importance to this work, we restate in modified form a theorem due to R. H. Bing [7].

THEOREM 1. *If* K *is a crumpled cube, then there exists a* 0*-dimensional* F_σ*-set* F *in* Bd K *such that* F $\cup$ Int K *is* 1-ULC. *Furthermore, if* $\{X_i\}$ *is a sequence of tame arcs in* Bd K, *then* F *may be chosen so that it lies in* (Bd K) - $\bigcup X_i$.

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The theorem above establishes the existence of F and gives some indication where F may lie in Bd K. It is important to note that F may not be unique.

Another important theorem for this work is the following, due to L. D. Loveland [20].

THEOREM 2. *Suppose* K *is a crumpled cube,* F *is a* 0-*dimensional* F_σ-*set in* Bd K *such that* F ∪ Int K *is* 1-ULC, *and* A *is a finite graph in* (Bd K) - F. *Then* A *is tame.*

We conclude this section with a useful lemma from the topology of E^2.

LEMMA 1. *Suppose* S *is a* 2-*sphere,* F *is a* 0-*dimensional* F_σ-*set in* S, C *is a closed set in* S - F, A *is a finite graph in* S, *and* $\varepsilon > 0$. *Then there exists a homeomorphism* f *on* A *such that* $f(A) \subset S - F$, $f \mid (C \cap A) = 1$, *and* $\rho(f, 1) < \varepsilon$.

3. THE MISMATCH THEOREM

The main result of this paper is the following theorem.

THEOREM 3. *Suppose* K_1 *and* K_2 *are crumpled cubes and* h *is a homeomorphism from* Bd K_1 *to* Bd K_2. *Then* $K_1 \cup_h K_2 \approx S^3$ *if and only if there exist disjoint* 0-*dimensional* F_σ-*sets* F_1 *and* F_2 *in* Bd K_1 *such that* $F_1 \cup \text{Int } K_1$ *and* $h(F_2) \cup \text{Int } K_2$ *are* 1-ULC.

Sufficiency. We assume that K_1 and K_2 are embedded in S^3 so that $S^3 - \text{Int } K_i$ is a 3-cell [17], [18]. Using Lemma 1, we find disks D_1 and D_2 in Bd K_1 such that

$$\text{Bd } K_1 = D_1 \cup D_2, \quad D_1 \cap D_2 = \text{Bd } D_1 = \text{Bd } D_2, \quad (F_1 \cup F_2) \cap \text{Bd } D_i = \emptyset.$$

By the hypothesis and Theorem 2, $h(\text{Bd } D_1)$ is tame. We push the interior of each disk $h(D_i)$ slightly into $S^3 - K_2$ to form a tame disk E_i [13] such that

$$\text{Bd } E_i = h(\text{Bd } D_i) \quad \text{and} \quad \text{Int } E_1 \cap \text{Int } E_2 = \emptyset.$$

The 2-sphere $E_1 \cup E_2$ is tame and thus bounds a cell C containing K_2; furthermore, $E_i \cup h(D_i)$ (i = 1, 2) bounds a cell C_i in C. There exists a homeomorphism g of C onto $S^3 - \text{Int } K_1$ such that

$$gh \mid \text{Bd } D_1 = 1, \quad g(E_1) = D_1, \quad g(E_2) = D_2.$$

Let U_i (i = 1, 2) be open sets such that $g(C_i) - \text{Bd } D_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$. The hypotheses of Theorem 4 are satisfied for the 3-cells $g(C_i)$ (i = 1, 2), the disks D_i, the homeomorphisms $gh \mid D_i$, the 0-dimensional, disjoint F_σ-sets $D_i \cap F_1$ and $D_i \cap F_2$, and the open sets U_i. The sufficiency of the condition in Theorem 3 will therefore be established when we have proved Theorem 4.

In the proof of the next theorem and of Lemma 2, we use the following concept.

Definition. A collection of 2-cells $D_1, \cdots, D_n$ in a disk D is a *cellular subdivision* of D if $\text{Int } D_i \cap \text{Int } D_j = \emptyset$ $(i \neq j)$ and $D = \bigcup D_i$. The *mesh* is the maximum of the numbers Diam $D_1, \cdots,$ Diam D_n.

THEOREM 4. *Suppose* C *is a* 3-*cell in* S^3, D *is a disk in* Bd C, h *is a homeomorphism of* D *onto* Cl((Bd C) - D) *such that* $h \mid \text{Bd } D = 1$, F_1 *and* F_2 *are disjoint* 0-*dimensional* F_σ-*sets in* Int D *such that* $F_1 \cup h(F_2) \cup \text{Ext } C$ *is* 1-ULC, *and*

U *is an open set containing* C - Bd D. *Then there exists a map* f *of* S^3 *onto* S^3 *such that*

(1) $f \mid S^3 - U = 1$,

(2) $f \mid S^3 - C$ *is a homeomorphism onto* $S^3 - f(C)$, *and*

(3) $f \mid D = fh$, *and* fh *is a homeomorphism onto* f(C).

Proof. We apply Lemma 2 repeatedly. First, we squeeze the 3-cell C to a finite collection of smaller 3-cell whose union is a disk with the 3-cells attached along subdisks. Each of these 3-cells we squeeze similarly to still smaller 3-cells. We continue the process, and in the limit C is entirely flattened to a disk. The details are as follows.

Let $\{U_1, U_2, \cdots\}$ be a sequence of open sets in S^3 such that $U \supset U_1 \supset U_2 \supset \cdots$ and $\bigcap U_i = C - \text{Bd } D$, and let $\{\varepsilon_1, \varepsilon_2, \cdots\}$ be a sequence of positive numbers such that $\sum_1^\infty \varepsilon_i < \infty$. The map f is the limit of a sequence of maps f_i defined inductively. Apply Lemma 2 to obtain a cellular subdivision $\{D_i^1\}$ of D of mesh less than ε_1 and a map f_1 of S^3 onto S^3 such that $f_1(D_i^1) \cup f_1 h(D_i^1)$ bounds a 3-cell K_i^1 in $f_1(C)$ of diameter less than ε_1. Let $\{V_i^1\}$ be a finite collection of disjoint open sets in $f_1(U_1)$ such that

$$K_i^1 - f_1(\text{Bd } D_i^1) \subset V_i^1 \quad \text{and} \quad \text{Diam } V_i^1 < \varepsilon_1 .$$

By induction, we can assume that $\{D_i^n\}$ is a cellular subdivision of D of mesh less than ε_n, f_n is a map of S^3 onto S^3, K_i^n is the 3-cell bounded by $f_n(D_i^n) \cup f_n h(D_i^n)$, and $\{V_i^n\}$ is a finite collection of disjoint open sets in $f_n(U_n)$ such that

$$K_i^n - f_n(\text{Bd } D_i^n) \subset V_i^n \quad \text{and} \quad \text{Diam } V_i^n < \varepsilon_n .$$

For each i, apply Lemma 2 to the 3-cell K_i^n, the disk $f_n(D_i^n)$, the homeomorphism $f_n h f_n^{-1} \mid f_n(D_i^n)$, the disjoint 0-dimensional F_σ-sets $f_n(D_i^n \cap F_1)$ and $f_n(D_i^n \cap F_2)$, the open set V_i^n, and use a sufficiently small ε to obtain a map f_{n+1}^i of S^3 and a cellular subdivision $\{E_j^i\}$ of $f_n(D_i^n)$ such that the mesh of $\{f_n^{-1}(E_j^i)\}$ is less than ε_{n+1} and the diameter of the 3-cell bounded by $f_{n+1}^i(E_j^i) \cup f_{n+1}^i f_n h f_n^{-1}(E_j^i)$ is less than ε_{n+1}. The mapping f_{n+1} consists of f_n followed by the mapping obtained by piecing together the mappings f_{n+1}^i. The collection $\{D_i^{n+1}\}$ is given by $\{f_n^{-1}(E_j^i)\}_{i,j}$, and each 2-sphere $f_{n+1}(D_i^{n+1}) \cup f_{n+1} h(D_i^{n+1})$ bounds a 3-cell K_i^{n+1} in $f_{n+1}(C)$. Choose a finite collection $\{V_i^{n+1}\}$ of disjoint open sets in $f_{n+1}(U_{n+1})$ such that

$$K_i^{n+1} - f_{n+1}(\text{Bd } D_i^{n+1}) \subset V_i^{n+1}, \quad \text{Diam } V_i^{n+1} < \varepsilon_{n+1}, \quad \text{and} \quad \bigcup_i V_i^{n+1} \subset \bigcup_i V_i^n .$$

It is easy to verify that the map $f = \lim f_i$ satisfies conditions (1), (2), and (3).

4. THE MAIN LEMMA

Definition. The arc A is a *spanning arc of the disk* D if $A \subset D$ and $A \cap \text{Bd } D = \text{Bd } A$. The disk D is a *spanning disk of the* 3-*cell* C if $D \subset C$ and $D \cap \text{Bd } C = \text{Bd } D$. The arc A is a *spanning arc of the annulus* H if $A \subset H$, $A \cap \text{Bd } H = \text{Bd } A$, and H - A is connected.

Let C be a 3-cell and D a disk in Bd C. We say that the cross-sectional diameter of C with respect to D is less than ε if there exists a homeomorphism g of $D \times I$ onto C such that $g(x, 0) = x$ for all $x \in D$ and Diam $g(D \times t) < \varepsilon$ for all $t \in I$.

LEMMA 2. *Suppose* C *is a* 3-*cell in* S^3, D *is a disk in* Bd C, h *is a homeomorphism of* D *onto* Cl((Bd C) - D) *such that* $h \mid \text{Bd } D = 1$, F_1 *and* F_2 *are disjoint* 0-*dimensional* F_σ-*sets in* Int D *such that* $F_1 \cup h(F_2) \cup \text{Ext } C$ *is* 1-ULC, U *is an open set containing* C - Bd D, *and* $\varepsilon > 0$. *Then there exist a cellular subdivision* $\{D_1, \cdots, D_n\}$ *of* D *with mesh less than* ε *and a map* f *of* S^3 *onto* S^3 *such that*

(1) $f \mid S^3 - U = 1$,

(2) $f \mid S^3 - C$ *is a homeomorphism onto* $S^3 - f(C)$,

(3) *both* $f \mid D$ *and* $f \mid h(D)$ *are homeomorphisms,*

(4) $\left(\bigcup \text{Bd } D_i\right) \cap (F_1 \cup F_2) = \emptyset$,

(5) $f(D) \cap fh(D) = f\left(\bigcup \text{Bd } D_i\right)$,

(6) $f \mid \bigcup \text{Bd } D_i = fh \mid \bigcup \text{Bd } D_i$, *and*

(7) $f(D_i) \cup fh(D_i)$ *bounds a* 3-*cell in* f(C) *of diameter less than* ε.

Proof. The proof consists of two main steps. In Step 1, C is squeezed to a finite collection of cross-sectionally small cells. In Step 2, each cell from Step 1 is squeezed to a finite collection of small cells. Step 2 is divided into three parts. In Part A, we use the 0-dimensional F_σ-set F_1 to obtain a special map that allows us in Part C to shorten the cells from Step 1. In Part B, we use the 0-dimensional F_σ-set $h(F_2)$ to achieve a partial splitting of the cells created in Step 1. In Part C, we shorten and split the cross-sectionally small cells, using the structures from Parts A and B. Parts A, B, and C are repeated in sequence a finite number of times, until the cells from Step 1 are sufficiently short.

Step 1. By B^2 we denote the standard unit square in E^2, by a the geometric center of B^2, by b a point in E^3 one unit below a, by B^3 the 3-cell that is the join of b and B^2, and by p the projection map of B^3 onto B^2 that moves points vertically. Let g be a homeomorphism of B^3 onto C such that $g^{-1}(D) = B^2$ and $pg^{-1}hg \mid B^2 = 1$. Using spanning arcs parallel to the edges of B^2, we partition B^2 into a finite collection $\{E_i\}$ of small squares. By Lemma 1, we may assume that

$$g(G) \cup hg(G) \subset \text{Bd } C - (F_1 \cup h(F_2)),$$

where $G = \bigcup \text{Bd } E_i$. It follows from Theorem 2 that the graph $g(G) \cup hg(G)$ is tame, and consequently, by [13], $g(p^{-1}(G))$ is tame. Corresponding to a point t in the line segment ab from a to b, let L(t) denote the join of t and Bd B^2, and if s and t are two points in ab, let L(s, t) denote the 3-cell in B^3 bounded by $L(s) \cup L(t)$. We also assume that Diam E_i is so small that Diam $g(p^{-1}(E_i) \cap L(t))$ is less than ε for all t in ab.

We complete Step 1 by pushing the tame graph hg(G) along $g(p^{-1}(G))$ to the graph g(G). Care must be taken, however, to insure that this squeezing of C produces cross-sectionally small cells. The required map is the composition of a finite collection $\{\alpha_i\}$ of maps of S^3 onto S^3, obtained as follows.

Let $b = t_0 < t_1 < \cdots < t_n = a$ be a partition of ab such that

$$\text{Diam}\ g(p^{-1}(E_i) \cap L(t_j, t_{j+1})) < \varepsilon .$$

For $i = 1, \cdots, n$, we let α_i denote the projection map of $T_i = g(p^{-1}(G) \cap L(t_{i-1}, t_i))$ onto $G_i = g(p^{-1}(G) \cap L(t_i))$ defined by the relation

$$\alpha_i(x) = g(p^{-1}(pg^{-1}(x)) \cap L(t_i)) .$$

We extend the maps α_i to S^3 one at a time, as follows. Select a small regular neighborhood V_1 of $T_1 - G_1$ such that $V_1 \cap g(L(t_1, t_n)) = \emptyset$. Extend the map α_1 to S^3 so that $\alpha_1 \mid S^3 - V_1 = 1$, so that $\alpha_1 \mid S^3 - T_1$ is a homeomorphism onto $S^3 - G_1$, and so that $\rho(\alpha_1, 1) < \varepsilon$. Let N_1 be a regular neighborhood of G in B^2, sufficiently near G to ensure that $\alpha_1 hg(N_1)$ lies in a thin tubular neighborhood of G_1. Let V_2 be a small regular neighborhood of $T_2 - G_2$ such that

$$V_2 \cap g(L(t_2, t_n)) = \emptyset \quad \text{and} \quad V_2 \cap \alpha_1 hg(B^2 - \text{Int}\ N_1) = \emptyset .$$

Now extend the map α_2 to S^3 so that $\alpha_2 \mid S^3 - V_2 = 1$, so that $\alpha_2 \mid S^3 - T_2$ is a homeomorphism onto $S^3 - G_2$, and so that $\rho(\alpha_2, 1) < \varepsilon$. Choose a regular neighborhood $N_2 \subset \text{Int}\ N_1$ of G in B^2 sufficiently close to G to ensure that $\alpha_2 \alpha_1 hg(N_2)$ lies in a thin tubular neighborhood of G_2. Continuing thus, we define $V_3, \alpha_3, N_3, V_4, \cdots, \alpha_n$.

The map squeezing C to a finite collection of cross-sectionally small cells is the composition $\alpha = \alpha_n \cdots \alpha_1$, and a typical cell is the 3-cell C^i bounded by $g(E_i) \cup \alpha hg(E_i)$. The cell C^i is cross-sectionally small with respect to $g(E_i)$, since for $j = 1, 2, \cdots, n - 1$ there exists a spanning disk of C^i near the disk $g(L(t_j) \cap p^{-1}(E_i))$. Figure 1 represents a schematic diagram of the squeezing process of Step 1.

Step 2. We squeeze the cross-sectionally small cells C^i from Step 1 to small cells. We find a finite collection $\{U^i\}$ of disjoint open sets in U such that $C^i - \text{Bd}\ g(E_i) \subset U^i$, and we squeeze each of the cells C^i individually, moving only points in U^i. For convenience, we drop the superscripts, identify the disk $g(E_i)$ with D, and use the notation and hypothesis of Lemma 2 with the additional requirement that the cross-sectional diameter of C with respect to D is less than ε.

We write C as the union of a finite collection $\{C_0, C_1, \cdots, C_n\}$ of 3-cells, each with diameter less than ε. We arrange the cells $\{C_i\}$ in a linear order such that $C_i \cap C_j = \emptyset$ if $|i - j| > 1$, and such that the set $C_{i-1} \cap C_i = \text{Bd}\ C_{i-1} \cap \text{Bd}\ C_i$ is a disk H_i. We select the cells $C_0, C_1, \cdots, C_n$ so that the disks $D, H_1, \cdots, H_n$ are disjoint and $D \subset \text{Bd}\ C_0$. By Lemma 1, we may assume that

$$h^{-1}\left(\bigcup \text{Bd}\ H_i\right) \cap (F_1 \cup F_2) = \emptyset ,$$

and since $F_1 \cup h(F_2) \cup \text{Ext}\ C$ is 1-ULC, we may assume that H_i and $h^{-1}(\text{Bd}\ H_i)$ are tame, by Theorem 2 and results of [13].

Part A. Denote by A the annulus $h^{-1}(\text{Bd}\ C_0)$, and by B the disk $D - \text{Int}\ A$. Push the interior of A into $\text{Int}\ C_0$ to form a tame annulus A' such that $\text{Bd}\ A' = \text{Bd}\ A$. Since $\text{Bd}\ B$ is tame, an improvement of the side-approximation theorem [19, Theorem 21] implies that there exist a finite collection of disjoint disks $I_1, \cdots, I_r$ in $\text{Int}\ B$, a null sequence of disjoint disks $I_{r+1}, I_{r+2}, \cdots$ in $\text{Int}\ A$, and a homeomorphism β_1 of D into $U \cup \text{Bd}\ D$ such that $\beta_1(D)$ is tame, $\beta_1 \mid \text{Bd}\ D = 1$,

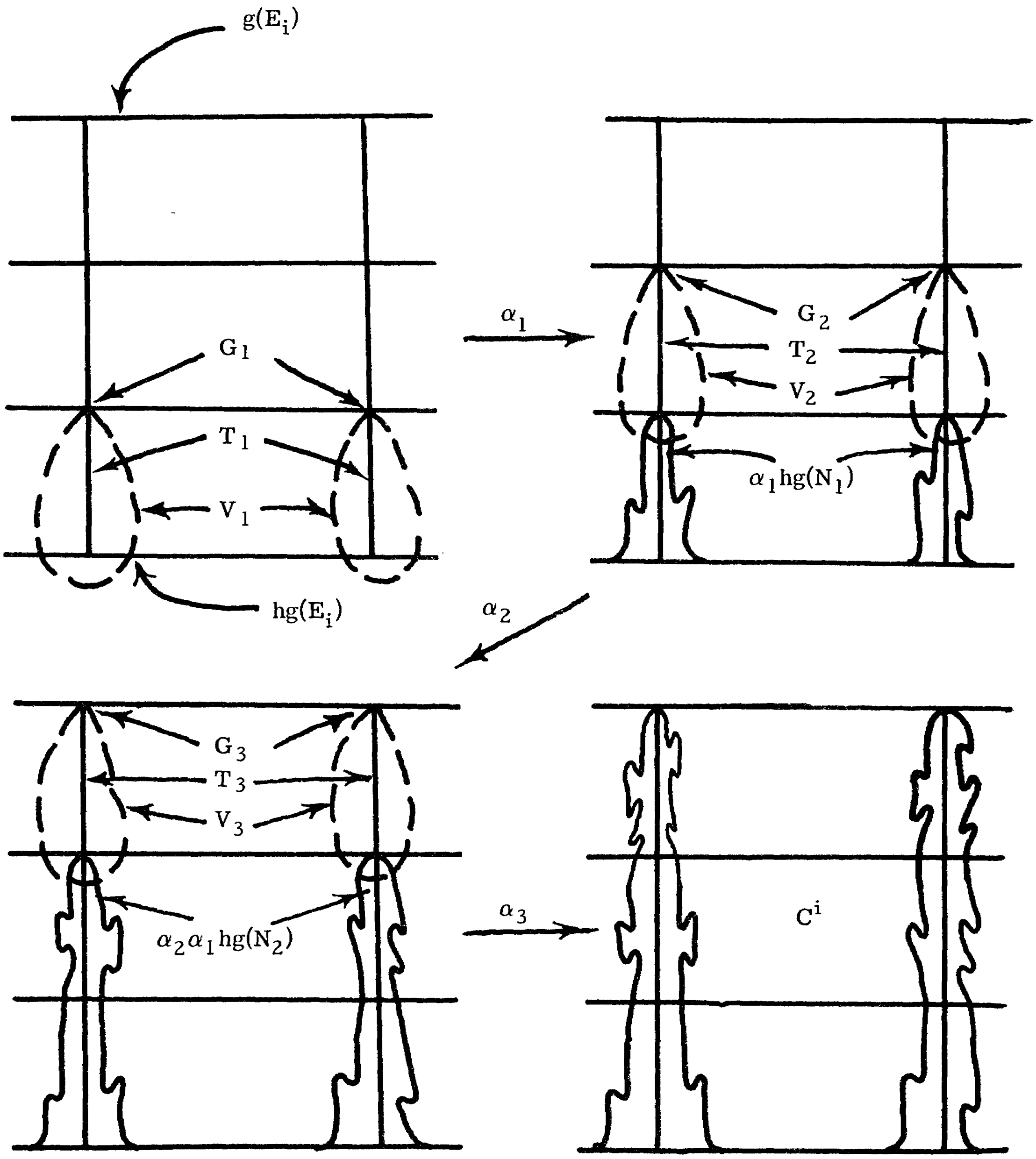

Figure 1.

$$\beta_1\left(\text{Int } D - \left(\bigcup \text{Int } I_i\right)\right) \subset \text{Ext } C, \quad \beta_1(I_i) \cap \text{Bd } C = \beta_1(I_i) \cap I_i,$$

and $\beta_1(D) \cap A' = \emptyset$. We also have control over the size of the loops $\beta_1(\text{Bd } I_i) \subset \text{Ext } C$ and over the distance through which β_1 moves points. Consequently, since $F_1 \cup h(F_2) \cup \text{Ext } C$ is 1-ULC, we may assume that the simple closed curve $\beta_1(\text{Bd } I_i)$ $(i = 1, \cdots, r)$ bounds a singular disk B_i in the intersection of $F_1 \cup \text{Ext } C$ and a small neighborhood of B. The set

$$\beta_1\left(D - \bigcup_{i=1}^{r} I_i\right) \cup \left(\bigcup_{i=1}^{r} B_i\right)$$

is the image of D under a map β_2 into $U + \text{Bd } D$. We constructed β_2 so that it possesses the following properties: $\beta_2 \mid \text{Bd } D = 1$, $\text{Diam } \beta_2(D) \cup C_0 < \varepsilon$, no singular point of β_2 is near Bd D, and if C' is the 3-cell in C bounded by $((\text{Bd } C) - A) \cup A'$, then $\beta_2(D) \cap C' = (\text{Bd } D) \cup K$, where K is a compact 0-dimensional subset of $F_1 \cap B$.

In Part C, we apply Dehn's lemma to the map β_2 and use the resulting disk to achieve a partial collapse of the 3-cell C_0. Points near the set K can not be moved very far; however, we have enough hypotheses to enable us to push the set h(K) to the cell C_0. The construction of a map of S^3 onto S^3 that pushes h(K) to H_1 and splits each of the cells $C_1, C_2, \cdots, C_n$ is the subject of Part B.

Part B. The compact 0-dimensional set K from Part A lies in $F_1 \cap B$, and since $F_1 \cap F_2 = \emptyset$, we can find arcs in $h(B - F_2)$ that contain h(K), by Lemma 1. In particular, we find a spanning disk E of the 3-cell $C_1 \cup C_2 \cup \cdots \cup C_n$ such that $H_1 \cap \text{Bd } E$ is a spanning arc A_1 of H_1, $h(B) \cap \text{Bd } E$ is a spanning arc A_2 of h(B), and $h(K) \subset A_2 \subset h(B - F_2)$. Since $\text{Bd } H_i \subset h(B - (K \cup F_2))$, we may choose E so that $E \cap C_i$ is a spanning disk of C_i $(i = 1, 2, \cdots, n)$. Since $F_1 \cup h(F_2) \cup \text{Ext } C$ is 1-ULC, Theorem 2 allows us to assume that E is tame. The simple closed curve $A_1 \cup A_2 = \text{Bd } E$ is the boundary of two disks R_1 and R_2 in the boundary of the cell $C_1 \cup \cdots \cup C_n$. The disk E splits the cell $C_1 \cup \cdots \cup C_n$ into two cells Q_1 and Q_2 such that $\text{Bd } Q_1 = E \cup R_1$ and $\text{Bd } Q_2 = E \cup R_2$. The cross-sectional diameter of Q_i is less than ε, and the length of Q_i is less than the length of C, in the sense that there are fewer terms in $(Q_i \cap C_1) \cup \cdots \cup (Q_i \cap C_n)$ than in $C_0 \cup C_1 \cup \cdots \cup C_n$.

We now find a map k_1 of S^3 onto S^3 that squeezes the disk E to the arc A_1. The existence of the map with the properties described below follows from the techniques of Step 1. Let V be a small regular neighborhood of $E - A_1$ such that $V \cap C_0 = \emptyset$.

There exists a map $k_1: S^3 \to S^3$ such that $k_1 \mid S^3 - V = 1$, such that $k_1 \mid S^3 - E$ is a homeomorphism onto $S^3 - A_1$, and such that $k_1 \mid E$ is a projection onto A_1. Furthermore, using the techniques of Step 1, we may choose k_1 so that the two 3-cells bounded by $k_1(R_1)$ and $k_1(R_2)$ are close approximations to the cells Q_1 and Q_2, respectively.

Using Lemma 1, we select a subdisk M of B such that $(\text{Bd } M) \cap (F_1 \cup F_2) = \emptyset$, the set $h^{-1}(A_2)$ is a spanning arc of M, $\text{Bd } M \cap \text{Bd } B = \text{Bd } h^{-1}(A_2)$, and $k_1 h(M)$ lies in a small tubular neighborhood W of A_1. With a homeomorphism $k_2: S^3 \to S^3$ that is the identity outside W, we push the boundary of $k_1 h(M)$ to H_1 in such a way that $k_2 k_1 h(\text{Bd } M)$ bounds a disk M' in H_1, so that A_1 is a spanning arc of M', and so that $k_2 k_1 h(\text{Bd } M) \cap \text{Bd } H_1 = \text{Bd } A_1$. Furthermore, we select a map k_2 so that

$$k_2 \mid k_1 h(\text{Bd } B) = 1, \quad \text{Int } C_0 \cap k_2 k_1(B) = \emptyset, \quad H_1 \cap k_2 k_1 h(B) = k_2 k_1 h(\text{Bd } B \cup \text{Bd } M).$$

In Part C, we use the map β_2 from Part A to achieve a partial collapse of the cell C_0 into a small neighborhood of H_1.

Part C. Let $B_i = h^{-1}(R_i) \cap (B - \text{Int } M)$ and $B_i' = (R_i \cap H_1) - \text{Int } M'$ $(i = 1, 2)$. Push the interiors of the disks B_1, M, and B_2 slightly into the interior of C_0 to form disks B_1'', M'', and B_2'' such that $\text{Bd } B_i = \text{Bd } B_i''$ and $\text{Bd } M = \text{Bd } M''$. Since $\text{Bd } D \cup \text{Bd } B \cup \text{Bd } M \subset D - F_1$, we may assume, by Theorem 2, that the disk $A' \cup B_1'' \cup M'' \cup B_2'' = D'$ is tame. We can now apply Dehn's lemma [22] to the singular disk $\beta_2(D)$ from Part A. Note that $\beta_2(D)$ fails to intersect B_1 and B_2, and that

$$\beta_2(D) \cap D' = \beta_2(\text{Bd } D) = \text{Bd } D = \text{Bd } D' ;$$

consequently, we may apply the lemma in a small enough neighborhood of $\beta_2(D)$ to ensure that the resulting disk H has the following properties: $H \cup D'$ is the boundary of a tame 3-cell X such that

$$B_i \subset X \quad (i = 1, 2), \qquad \text{Diam } X \cup C_0 < \varepsilon, \qquad X \subset U \cup \text{Bd } D.$$

We need the following notation, in order to describe a map q that partially collapses C_0. Let H' be an annulus in the disk H such that $\text{Bd } H \subset \text{Bd } H'$, and let A'' be the annulus obtained by pushing the interior of the annulus $A' \cup H'$ slightly into the interior of X. The annulus A'' is constructed so that $A' \cup A'' \cup H'$ bounds a solid torus $Y \subset X$ in such a way that $C(X - Y)$ is a 3-cell Z with $B_i \subset Z$ $(i = 1, 2)$. Push the interior of the disk H_1 slightly into Int C_0 to form a disk H_1' such that $H_1' \cup H_1$ bounds a tame 3-cell Z' with the property that $\text{Diam } Z' \cup C_1 < \varepsilon$. Let H_1'' be an annulus in H_1' such that $\text{Bd } H_1' \subset \text{Bd } H_1''$, and let H'' be the annulus obtained by pushing the interior of the annulus $h(A) \cup H_1''$ into the interior of the 3-cell $C_0 - \text{Int } Z'$. The annulus H'' is constructed so that $h(A) \cup H_1'' \cup H''$ bounds a solid torus $Y' \subset C_0$.

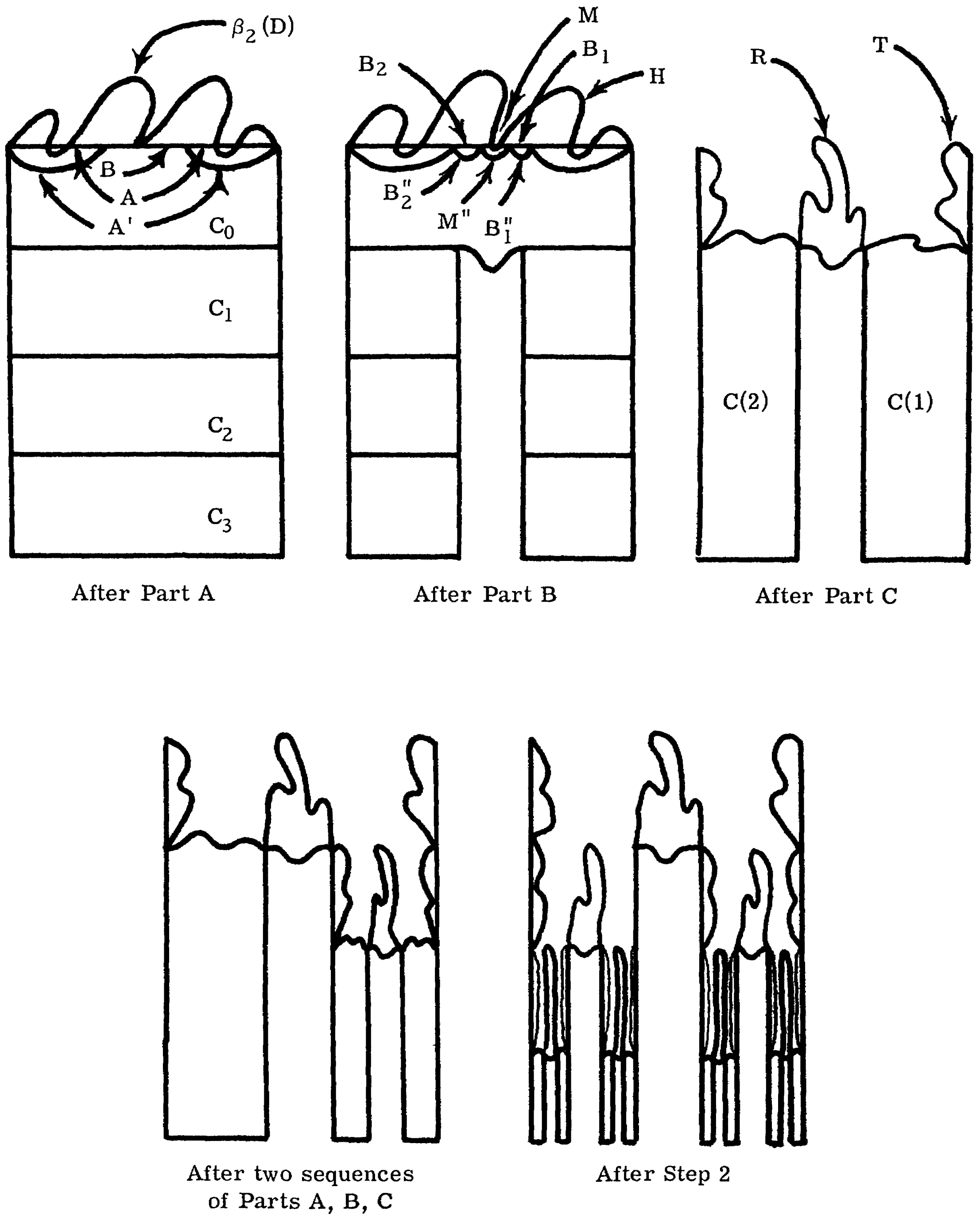

Figure 2.

Let U' be a thin shell neighborhood of Int H. A mapping q of S^3 onto S^3 may now be described as follows: $q \mid S^3 - (C \cup X \cup U') = 1$, $q \mid X \cup U'$ is a homeomorphism into $C_0 \cup X \cup U'$, $q(Y) = Y'$, $q(Z) = Z'$, and

$$q \mid \text{Bd } D \cup \text{Bd } B \cup \text{Bd } M = k_2 k_1 h \mid \text{Bd } D \cup \text{Bd } B \cup \text{Bd } M .$$

We place no restriction on the extension of q to the remainder of S^3.

We observe that the diameter of the 3-cell R bounded by $q(M) \cup k_2 k_1 h(M)$ is less than ε, since Diam $C_0 \cup X < \varepsilon$ and $k_2 k_1 h(M)$ lies in a small tubular neighborhood of $A_1 \subset \text{Bd } C_0$; that the solid torus T bounded by $q(A) \cup h(A)$ has diameter less than ε, since Diam $C_0 \cup X < \varepsilon$; and that for $i = 1, 2$, the 3-cell C(i) bounded by $q(B_i) \cup k_2 k_1 h(B_i)$ has cross-sectional diameter no greater than that of C, but is shorter in the sense that one fewer cell of diameter ε is required to describe it linearly.

Finally, let J_1 and J_2 be disjoint spanning arcs of A such that $J_i \subset A - (F_1 \cup F_2)$. We push the arcs $q(J_1)$ and $q(J_2)$ to the arcs $h(J_1)$ and $h(J_2)$, respectively, along disjoint tame meridional disks of T. The image of T consists of two 3-cells, each with diameter less than ε.

Let U_1 and U_2 be disjoint open sets containing $C(1) - q(\text{Bd } B_1)$ and $C(2) - q(\text{Bd } B_2)$, respectively. We repeat Parts A, B, and C in sequence for each of the pairs $(C(1), U_1)$ and $(C(1), U_2)$, and again for the four resulting pairs. The process continues until all cross-sectionally small cells have been shortened to size ε. Figure 2 represents a schematic diagram of Parts A, B, and C.

5. NECESSITY

In this section we reduce the proof of the necessity in Theorem 3 to the following two-sided-approximation theorem established in [14] (notation: $S = \text{Bd } K_1 = h(\text{Bd } K_1) = \text{Bd } K_2$).

THEOREM 5. *If* S *is a* 2-*sphere in* E^3 *and* $\varepsilon > 0$, *then there exist a finite collection* $\{D^1, \cdots, D^n, E^1, \cdots, E^n\}$ *of disjoint* ε-*disks in* S *and* ε-*homeomorphisms* $f, g: S \to E^3$ *such that*

(1) $f\left(S - \bigcup \text{Int } D^i\right) \subset \text{Int } S$,

(2) $f(D^i) \cap S \subset \text{Int } D^i$,

(3) $g\left(S - \bigcup \text{Int } E^i\right) \subset \text{Ext } S$, *and*

(4) $g(E_i) \cap S \subset \text{Int } E^i$.

Furthermore, we may assume that f(S) *and* g(S) *are polyhedral and* $f(S) \cap g(S) = \emptyset$.

We take the disjoint 0-dimensional F_σ-sets required in Theorem 3 to be sums of intersections of unions of disks obtained by applying Theorem 5 repeatedly. Let $\varepsilon_1 = 1$, apply Theorem 5 with $\varepsilon = \varepsilon_1$, and let D_1^i, E_1^i, f_1, g_1 be the resulting disjoint ε_1-disks and ε_1-homeomorphisms. Let D_1^{i*} and E_1^{i*} be subdisks of Int D_1^i and Int E_1^i, respectively, such that

$$D_1^i \cap f_1(D_1^i) \subset \text{Int } D_1^{i*} \quad \text{and} \quad E_1^i \cap g_1(E_1^i) \subset \text{Int } E_1^{i*} .$$

Proceeding inductively, we assume that ε_{n-1}, D^i_{n-1}, E^i_{n-1}, f_{n-1}, g_{n-1}, $D^i_{n-1}{}^*$ and $E^i_{n-1}{}^*$ have been defined. Let ε_n be a positive number less than each of the numbers

$$(1/8)\varepsilon_{n-1}, \quad (1/4)\rho(S - D^i_{n-1}, D^i_{n-1}{}^*), \quad (1/4)\rho(S - E^i_{n-1}, E^i_{n-1}{}^*),$$
$$\rho(S - D^i_{n-1}{}^*, f_{n-1}(D^i_{n-1})), \quad \rho(S - E^i_{n-1}{}^*, g_{n-1}(E^i_{n-1})),$$
$$\rho(D^i_{n-1}, f_{n-1}(S - D^i_{n-1})), \quad \rho(E^i_{n-1}, g_{n-1}(S - E^i_{n-1})).$$

Apply Theorem 5 with $\varepsilon = \varepsilon_n$, and let D^i_n, E^i_n, f_n, g_n be the resulting disjoint ε_n-disks and ε_n-homeomorphisms. As above, let $D^i_n{}^*$ and $E^i_n{}^*$ be subdisks of Int D^i_n and Int E^i_n, respectively, such that

$$D^i_n \cap f_n(D^i_n) \subset \text{Int } D^i_n{}^* \quad \text{and} \quad E^i_n \cap g_n(E^i_n) \subset \text{Int } E^i_n{}^*.$$

Let

$$F = \bigcup_{k=1}^{\infty}\left(\bigcap_{n=k}^{\infty}\left(\bigcup_i D^i_n\right)\right) \quad \text{and} \quad G = \bigcup_{k=1}^{\infty}\left(\bigcap_{n=k}^{\infty}\left(\bigcup_i E^i_n\right)\right).$$

The sets F and G are clearly 0-dimensional F_σ-sets. They are disjoint, since $\left(\bigcup_i D^i_n\right) \cap \left(\bigcup_i E^i_n\right) = \emptyset$ for each n.

Before we show that $F \cup \text{Int } S$ is 1-ULC, we establish the existence of a special map β_n taking the 3-cell $B_n = f_n(S) \cup \text{Int } f_n(S)$ to the 3-cell

$$B_{n+1} = f_{n+1}(S) \cup \text{Int } f_{n+1}(S).$$

Let B^i_n be the component of $B_n - f_{n+1}(D^i_n{}^*)$ that contains $f_n(S - D^i_n)$. By Tietze's extension theorem, there exists a map

$$\beta^i_n \colon B_n \to B^i_n \cup f_{n+1}(D^i_n{}^*)$$

such that $\beta^i_n \mid B^i_n = 1$ and $\beta^i_n(B_n - B^i_n) \subset f_{n+1}(D^i_n{}^*)$. We obtain the map β_n by piecing together the finite collection of maps β^1_n, β^2_n, $\cdots$. The diameter of the set $(B_n - B^i_n) \cup D^i_n{}^*$ is less than $3\varepsilon_n$, and since $8\varepsilon_{n+1} < \varepsilon_n$, the diameter of the set $U^i_n = N((B_n - B^i_n) \cup D^i_n{}^*, 4\varepsilon_{n+1})$ is less than $4\varepsilon_n$. It is a simple exercise to verify that the maps $\beta_n \colon B_n \to B_{n+1}$ and the sets U^i_n have the properties

(5) $\beta_n \mid B_n - \bigcup_i U^i_n = 1$,

(6) $\beta_n(U^i_n \cap B_n) \subset U^i_n$,

(7) $D^j_{n+1} \subset D^i_n$ and $U^j_{n+1} \subset U^i_n$ if $\beta_n(U^i_n \cap B_n) \cap U^j_{n+1} \neq \emptyset$, and

(8) $B_n - \bigcup_i U^i_n \subset \text{Int } S$.

By the argument in [7, Theorem 4.2], it will follow that $F \cup \text{Int } S$ is 1-ULC provided to each positive number ε there corresponds an integer k with the property that for each $n \geq k$, there exists an ε-map α_n such that α_n takes the 3-cell B_n into $F \cup \text{Int } S$ and $\alpha_n \mid B_n - N(\text{Bd } B_n, \varepsilon) = 1$. For a fixed positive ε, take k so large that $4\varepsilon_k < \varepsilon$. If $n \geq k$, let $\alpha_n = \cdots \beta_{n+1} \beta_n$. Since

$$\text{Diam } U_n^i < 4\varepsilon_n < 4\varepsilon_k < \varepsilon ,$$

it follows by (5), (6), and (7) that α_n moves no points further than ε and that α_n is the identity outside $\bigcup_i U_n^i \subset N(\text{Bd } B_n, \varepsilon)$. Furthermore, by (5), (6), (7), and (8), we see that for each $x \in B_n$ the sequence $\{\beta_n(x), \beta_{n+1}\beta_n(x), \cdots\}$ either eventually has a constant value in Int S or eventually is in each set of a chain $U_n^{i_n} \supset U_{n+1}^{i_{n+1}} \supset \cdots$ for which $D_n^{i_n} \supset D_{n+1}^{i_{n+1}} \supset \cdots$. In either case, $\alpha_n(x) \in F \cup \text{Int } S$, since

$$\bigcap_{j=n}^{\infty} U_j^{i_j} = \bigcap_{j=n}^{\infty} D_j^{i_j} \in F .$$

It now follows that $F \cup \text{Int } S$ is 1-ULC. Similarly, $G \cup \text{Ext } S$ is 1-ULC.

6. SOME COROLLARIES

THEOREM 6. *Suppose* K_1 *and* K_2 *are crumpled cubes and* h *is a homeomorphism of* $\text{Bd } K_1$ *to* $\text{Bd } K_2$, *and choose* $\varepsilon > 0$. *Then there exists a homeomorphism* g *of* $\text{Bd } K_1$ *to* $\text{Bd } K_2$ *such that* $\rho(g, h) < \varepsilon$ *and* $K_1 \cup_g K_2 \approx S^3$.

(This is the main result of [10].)

Proof. By Theorem 1, there exist 0-dimensional F_σ-sets F_1 and F_2 in $\text{Bd } K_1$ and $\text{Bd } K_2$, respectively, such that $F_1 \cup \text{Int } K_1$ and $F_2 \cup \text{Int } K_2$ are 1-ULC. Let $\{A_1, A_2, \cdots\}$ be a sequence of arcs in $\text{Bd } K_1$ covering F_1. Using Lemma 1, we can push the arcs $h(A_i)$ one at a time into $(\text{Bd } K_2) - F_2$ (see [6, Theorem 7]); hence, there exists a homeomorphism t of $\text{Bd } K_2$ to $\text{Bd } K_2$ such that $\rho(t, 1) < \varepsilon$ and $t\left(\bigcup h(A_i)\right) \cap F_2 = \emptyset$. The homeomorphism $g = th$ mismatches F_1 and F_2; hence $K_1 \cup_g K_2 \approx S^3$, by Theorem 3.

J. R. Stallings [23] gives an example of a crumpled cube T in which there is a Cantor set of nonpiercing points in Bd T.

THEOREM 7. *Suppose* T *is the crumpled cube given by Stallings* [23], W *is the set of points in* Bd T *where* T *fails to be a* 3*-manifold with boundary,* K *is a crumpled cube, and* h *is a homeomorphism from* Bd T *to* Bd K. *Then* $T \cup_h K \approx S^3$ *if and only if* $K - h(W)$ *is* 1-ULC.

Proof. Suppose $T \cup_h K \approx S^3$. By Theorem 3, there exist disjoint 0-dimensional F_σ-sets F and G in Bd T such that $F \cup \text{Int } T$ and $h(G) \cup \text{Int } K$ are 1-ULC. Since each point of W is a nonpiercing point, it follows by D. R. McMillan's characterization [21] of piercing points that $W \subset F$. Loops in $K - h(W)$ can be pushed slightly into Int K without intersecting $h(W)$; hence, $K - h(W)$ is 1-ULC, since $h(W) \cap (h(G) \cup \text{Int } K) = \emptyset$ and $h(G) \cup \text{Int } K$ is 1-ULC.

Conversely, if $K - h(W)$ is 1-ULC, then there exists a 0-dimensional F_σ-set $G \subset (\text{Bd } T) - W$ such that $h(G) \cup \text{Int } K$ is 1-ULC. Since W is the set of points in T where T fails to be a 3-mainfold, $W \cup \text{Int } T$ is 1-ULC. By Theorem 3, $T \cup_h K \approx S^3$.

Definition. A crumpled cube K is *universal* if for each crumpled cube K' and each homeomorphism h of Bd K to Bd K', the space $K \cup_h K'$ is homeomorphic to S^3.

The concept of a universal crumpled cube was introduced and studied by R. J. Daverman and W. T. Eaton [12] before the techniques of this paper were available. The results of this section completely solve the research problems discussed near the end of [12].

The following theorem characterizes universal crumpled cubes. C. D. Bass and R. J. Daverman have used decomposition-space techniques to give an independent proof of the necessity.

THEOREM 8. *A crumpled cube* K *is universal if and only if for each Cantor set* C *in* Bd K, *the set* K - C *is* 1-ULC.

Proof. Suppose K - C is 1-ULC for each Cantor set $C \subset$ Bd K. Let K' be a crumpled cube, and let h be a homeomorphism of Bd K to Bd K'. By Theorem 1, there exists a 0-dimensional F_σ-set F' in K' such that $F' \cup$ Int K' is 1-ULC. Let $F' = C_1 \cup C_2 \cup \cdots$, where C_i is compact and 0-dimensional. Since the set $K - h^{-1}(C_i)$ is 1-ULC, compact, and 0-dimensional, it follows from the techniques of [7] or [8] that $h^{-1}(C_i)$ lies on a tame arc A_i. By Theorem 1, there exists a 0-dimensional F_σ-set F in (Bd K) - $\left(\bigcup A_i\right)$ such that $F \cup$ Int K is 1-ULC. The homeomorphism h mismatches the sets F and F'; hence $K \cup_h K' \approx S^3$, by Theorem 3.

Conversely, if there exists a Cantor set $C \subset$ Bd K such that K - C is not 1-ULC, then by Theorem 7 the sum $T \cup_h K$ is not S^3, where T is the crumpled cube of Stallings [23] and h is a homeomorphism of Bd T to Bd K sending the non-manifold points W of Bd T onto C.

COROLLARY 1. *A crumpled cube* K *is universal if each arc in* Bd K *is tame.*

It is known that the arcs in the boundaries of the crumpled cubes described by R. H. Bing [5] and D. S. Gillman [15] are tame.

COROLLARY 2. *The crumpled cubes of Bing* [5] *and Gillman* [15] *are universal.*

The crumpled cubes described by W. R. Alford [1] may have wild arcs in their boundaries; however, the Cantor sets in their boundaries satisfy the condition in Theorem 8.

COROLLARY 3. *The crumpled cubes of Alford* [1] *are universal.*

The following theorem characterizes the sums of crumpled cubes that are topologically equivalent to S^3, provided one of the crumpled cubes is the solid Alexander horned sphere in [4] or in Figure 3.

THEOREM 9. *Suppose* H *is the solid Alexander horned sphere,* W *is the set of points in* Bd H *where* H *fails to be a* 3-*manifold with boundary,* K *is a crumpled cube, and* h *is a homeomorphism from* Bd H *to* Bd K. *Then* $H \cup_h K \approx S^3$ *if and only if there exists a countable dense subset* F *of* W *such that* h(p) *is a piercing point of* K *for each* $p \in F$.

Proof. Suppose $H \cup_h K \approx S^3$. Then, by Theorem 3, there exist disjoint 0-dimensional F_σ-sets D and E in Bd H such that $D \cup$ Int H and $h(E) \cup$ Int K are 1-ULC. The set D must be dense in W; hence there exists a countable set $F \subset D$ that is also dense in W. For $p \in F$, h(p) belongs to (Bd K) - h(E), and we can push loops in (Bd K) - h(p) to Int K without intersecting h(p). Hence K - h(p) is 1-ULC, since $h(E) \cup$ Int K is 1-ULC. By [21], h(p) is a piercing point.

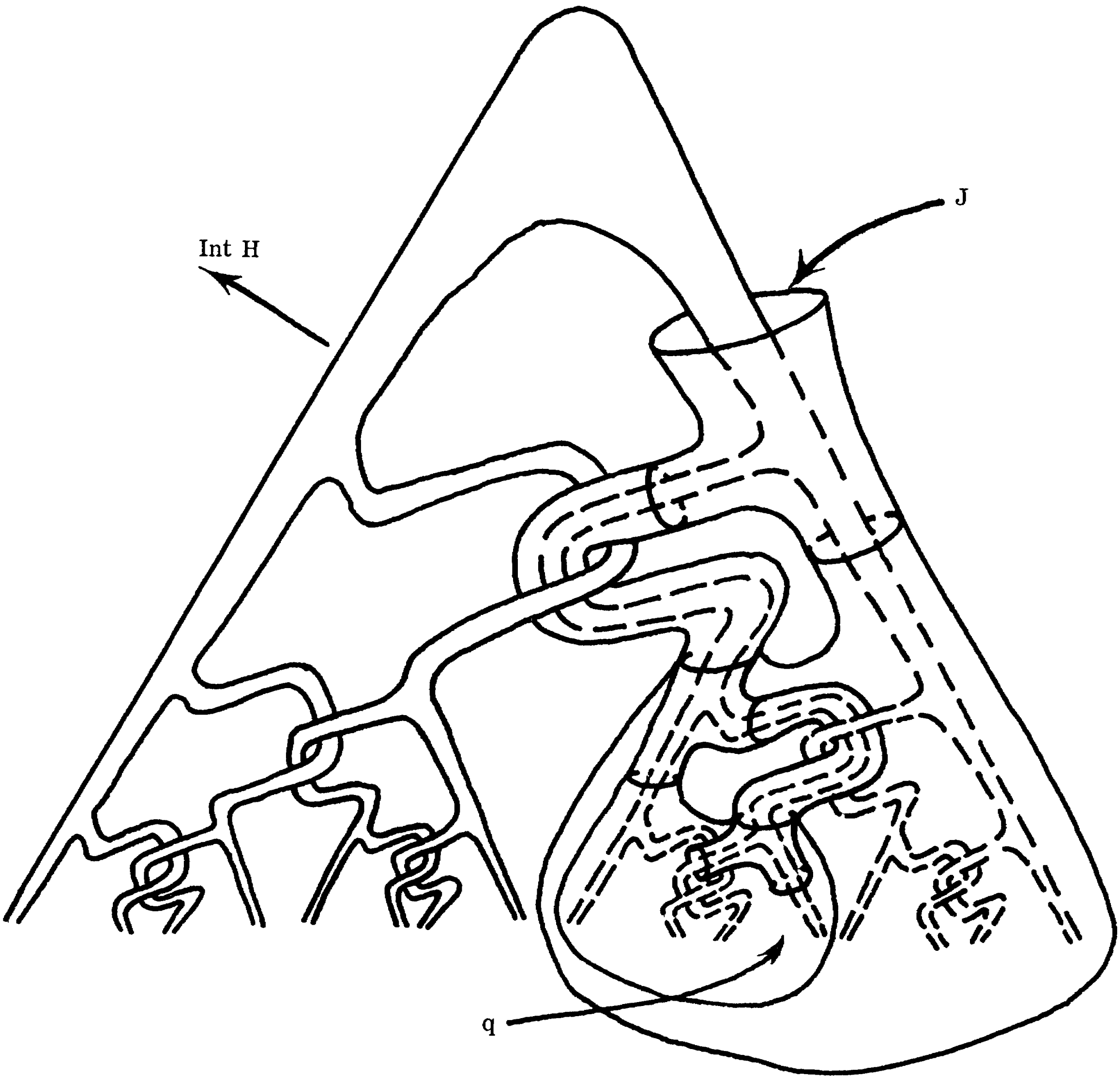

Figure 3.

Conversely, let F be a countable dense subset of W. In Figure 3, loop J is shrunk to a point in $\{q\} \cup \text{Int } H$, where $q \in W$. The reader may easily extend the illustrated horn-switching technique to show that $F \cup \text{Int } H$ is 1-ULC; the result also follows by the techniques of [15]. If h(p) is a piercing point of K for each $p \in F$, then, since F is countable, it follows from techniques of [7] or [8] that there exists a 0-dimensional F_σ-set $G \subset (\text{Bd } K) - h(F)$ such that $G \cup \text{Int } K$ is 1-ULC. By Theorem 3, $H \cup_h K \approx S^3$.

Definition. A crumpled cube H is *self-universal* if $H \cup_h H \approx S^3$ for each homeomorphism h of Bd H to itself.

By using decomposition-space techniques, B. G. Casler [9] has shown that the solid Alexander horned sphere H is self-universal. Bass and Daverman [3] have shown that H is not universal, by exhibiting a special upper-semicontinuous decomposition of S^3 that is not S^3. These results also follow from Theorem 7 and Theorem 9, since each point in Bd H is a piercing point of H. Some of the other theorems about decomposition spaces that are corollaries to Theorem 3 are presented in [15].

REFERENCES

1. W. R. Alford, *Some "nice" wild 2-spheres in* E^3. Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), pp. 29-33. Prentice Hall, Englewood Cliffs, N. J., 1962.
2. S. Armentrout, *Concerning cellular decompositions of 3-manifolds that yield 3-manifolds.* Trans. Amer. Math. Soc. 133 (1968), 307-332.
3. C. D. Bass and R. J. Daverman, *A self-universal crumpled cube which is not universal.* Bull. Amer. Math. Soc. 76 (1970), 740-742.
4. R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres.* Ann. of Math. (2) 56 (1952), 354-362.
5. ———, *A wild surface each of whose arcs is tame.* Duke Math. J. 28 (1961), 1-15.
6. ———, *Each disk in* E^3 *contains a tame arc.* Amer. J. Math. 84 (1962), 583-590.
7. ———, *Pushing a 2-sphere into its complement.* Michigan Math. J. 11 (1964), 33-45.
8. J. W. Cannon, *Sets which can be missed by side approximations to spheres.* Pacific J. Math. 34 (1970), 321-334.
9. B. G. Casler, *On the sum of two solid Alexander horned spheres.* Trans. Amer. Math. Soc. 116 (1965), 135-150.
10. R. J. Daverman and W. T. Eaton, *A dense set of sewings of two crumpled cubes yields* S^3. Fund. Math. 65 (1969), 51-60.
11. ———, *The Cartesian product of a line with the union of two crumpled cubes.* Notices Amer. Math. Soc. 15 (1968), 542, #68T-349.
12. ———, *Universal crumpled cubes.* Topology (to appear).
13. P. H. Doyle and J. G. Hocking, *Some results on tame disks and spheres in* E^3. Proc. Amer. Math. Soc. 11 (1960), 832-836.
14. W. T. Eaton, *A 2-sided approximation theorem for 2-spheres* (to appear).
15. ———, *Applications of a mismatch theorem to decomposition spaces* (to appear).
16. D. S. Gillman, *Note concerning a wild sphere of Bing.* Duke Math. J. 31 (1964), 247-254.
17. N. Hosay, *The sum of a cube and a crumpled cube is* S^3. Notices Amer. Math. Soc. 11 (1964), Errata for Volume 10, 152.
18. L. L. Lininger, *Some results on crumpled cubes.* Trans. Amer. Math. Soc. 118 (1965), 534-549.
19. F. M. Lister, *Simplifying intersections of disks in Bing's side approximation theorem.* Pacific J. Math. 22 (1967), 281-295.
20. L. D. Loveland, *Tame subsets of spheres in* E^3. Pacific J. Math. 19 (1966), 489-517.
21. D. R. McMillan, Jr., *Some topological properties of piercing points.* Pacific J. Math. 22 (1967), 313-322.

22. Ch. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*. Ann. of Math. (2) 66 (1957), 1-26.

23. J. R. Stallings, *Uncountably many wild disks*. Ann. of Math. (2) 71 (1960), 185-186.

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