HYPERINVARIANT SUBSPACES FOR OPERATORS ON THE SPACE OF COMPLEX SEQUENCES

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Let (s) denote the space of all complex sequences (functions on the positive integers) with the seminorms

$$\|f\|_n = \max_{1 \leq j \leq n} |f(j)| \qquad (n = 1, 2, \cdots).$$

By an operator on (s) we mean a continuous linear transformation of (s) into itself; by a subspace of (s) we mean a closed vector subspace. A subspace is said to be *invariant* for an operator if it is mapped into itself by the operator, and *hyperinvariant* (see [1]) if it is invariant for every operator commuting with the given operator. In this note we show that to each operator on (s) that is not a scalar multiple of the identity operator, there corresponds a proper hyperinvariant subspace. This answers a question raised in [3].

Notation. By $\mathbb{C}$ we denote the complex field, and by (s_0) the space of all sequences of complex numbers that have only finitely many nonzero elements. Thus (s_0) is a vector space of dimension $\aleph_0$ over $\mathbb{C}$.

There is a duality between (s) and (s_0):

$$(f, p) = \sum f(n)\, p(n) \qquad (f \in (s),\ p \in (s_0)). \tag{1}$$

Each p induces a continuous linear functional on (s), and every continuous linear functional has this form. Further, each f induces an algebraic linear functional on (s_0), and every algebraic linear functional has this form. The space (s) and the space (s_0) with its strong dual topology, that is, the topology of uniform convergence on bounded subsets of (s), are dual spaces. Every linear transformation on (s_0) is continuous, and every linear subspace in (s_0) is closed.

If S is a vector subspace of (s_0), then $S^\perp$ denotes the annihilator of S in (s). This is always a closed subspace, and it is proper if and only if S is proper.

THEOREM 1. *Every operator on* (s) *that is not a scalar multiple of the identity has a proper hyperinvariant subspace.*

Proof. Let U be an operator on (s). Because of the duality between (s) and (s_0), it will be sufficient to show that the adjoint transformation U^* on (s_0) has a proper hyperinvariant subspace; the annihilator of this subspace will be the desired subspace for U. The following lemma and its corollary will complete the proof.

LEMMA 1. *Every algebraic linear transformation on* (s_0) *has nonempty spectrum.*

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(The spectrum of an operator on (s_0) is understood in the purely algebraic sense: λ is in the spectrum of T if no algebraic linear transformation on (s_0) inverts $T - \lambda$.)

Proof of Lemma 1. Let T be a linear transformation on (s_0). We must show that there exists a complex number λ such that $T - \lambda$ (we write $T - \lambda$ in place of $T - \lambda I$) is not invertible.

Assume that no such λ exists, so that $T - \lambda$ is always invertible. Then every nonzero polynomial

$$p(T) = c(T - \lambda_1) \cdots (T - \lambda_n)$$

is invertible. Hence all nonzero rational functions $p(T)[q(T)]^{-1}$ are invertible.

Let R denote the field of all rational functions, regarded as a vector space over $\mathbf{C}$, and choose any vector f in (s_0) (not the zero vector). We define a map from R to (s_0) by the formula

$$r \mapsto r(T)f \qquad (r \in R).$$

This is a linear transformation, and it is one-to-one. Indeed, if $r(T)f$ were 0, then $r(T)$ would not be invertible, which is impossible (unless $r = 0$). But now we have a contradiction, since $\dim (s_0) = \aleph_0$, whereas $\dim R > \aleph_0$ (for example, the functions $\{r_\lambda(z) = (z - \lambda)^{-1}: \lambda \in \mathbf{C}\}$ are linearly independent). This completes the proof.

COROLLARY. *Every linear transformation on* (s_0) *that is not a scalar multiple of the identity has a hyperinvariant subspace.*

Proof. Let T be a linear transformation. By the lemma, there exists a complex number λ such that $T - \lambda$ is not invertible. Then either $T - \lambda$ fails to be one-to-one, or it is not surjective. In the first case, the kernel of $T - \lambda$ is a proper subspace (it cannot be all of (s_0), since $T \neq \lambda I$) which is easily seen to be hyperinvariant for T. In the second case, the range of T is a proper hyperinvariant subspace. ■

The assertion of Lemma 1 is not valid for all vector spaces over $\mathbf{C}$, nor is it valid for vector spaces of dimension $\aleph_0$ over arbitrary algebraically closed fields. The latter fact was pointed out to us by J. E. McLaughlin. We require a lemma.

LEMMA 2. *If* F *is a field and* E *a subfield, and if* $f \in F \setminus E$, *then the linear transformation* M_f *of multiplication by* f *on* F (*where* F *is regarded as a vector space over* E) *has empty spectrum.*

Proof. We must show that the operator $M_f - eI$ is invertible, for each $e \in E$. The operator represents multiplication by $f - e$, and this is invertible since F is a field and $f - e \neq 0$.

COROLLARY 1. *There exist a vector space* V *over* $\mathbf{C}$ *and a linear transformation on* V *with empty spectrum.*

COROLLARY 2. *If* A *is the field of algebraic numbers and* A_0^∞ *is the vector space of dimension* $\aleph_0$ *over* A, *then there exists a linear transformation on* A_0^∞ *with empty spectrum.*

Proof. Let R_A denote the field of rational functions over A. By considering the partial-fractions decomposition, we see that R_A, regarded as a vector space over A, has dimension $\aleph_0$. The result now follows from Lemma 2.

THEOREM 2. *Every operator on* (s) *has closed range.*

Proof. If we give (s_0) the weak* topology resulting from the pairing (1), then every subspace is closed. Thus, if T is an operator on (s), then its adjoint has weak* closed range. But by a general theorem on F-spaces, this implies that T has closed range (see [2, Chapter IV, Theorem 7.7, p. 160]). ∎

Every linear transformation on (s_0) can be represented by a *column-finite* infinite matrix $T = (t_{nm})$ $(n, m = 1, 2, \cdots)$ (that is, by a matrix each of whose columns has only finitely many nonzero elements). If all the elements t_{nm} are algebraic numbers, then T is also a linear transformation on A_0^∞.

COROLLARY. *There exists a column-finite matrix whose elements are algebraic numbers but whose spectrum is nonvoid and contains only transcendental numbers.*

Proof. Choose the matrix representing the linear transformation T of Corollary 2 to Lemma 2. This matrix has empty spectrum as an operator on A_0^∞; but by Lemma 1, the spectrum as an operator on (s_0) is nonempty. We claim that if λ is in the spectrum, then λ is transcendental.

Indeed, let λ be an algebraic number, and let $S = T - \lambda$. Then S is invertible as an operator on A_0^∞. Hence its range is all of A_0^∞. Hence its range, as an operator on (s_0), must contain the vector subspace of (s_0) spanned by A_0^∞. Since A_0^∞ contains the standard basis vectors, this subspace is all of (s_0), that is, S is surjective.

Since by formula (1) the algebraic dual of A_0^∞ can be identified with A^∞ and the algebraic adjoint of the restriction of S to A_0^∞ is the restriction of S^* to A^∞, the transformation S^* is invertible as an operator on A^∞. Hence the range of S^* is all of A^∞. Its range, as an operator on (s), is closed (Theorem 2) and contains the dense subset A^∞; therefore, S^* is surjective. Hence S (as an operator on (s_0)) is one-to-one. Thus S is invertible, and therefore λ is not in the spectrum of T. ∎

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