THE CYCLIC CONNECTIVITY OF PLANE CONTINUA

Charles L. Hagopian

Suppose that p and q are distinct points of a locally connected, compact, plane continuum H. It is known that if no point separates p from q in H, then there exists a simple closed curve in H that contains both p and q [5]. It is also known that H is arcwise connected [4, Theorem 13, p. 91]. Recently, arcwise connectedness has been established for certain plane continua that are not locally connected [1]. A continuum M is said to be *aposyndetic at a point* p *of* M *with respect to a set* N *in* $M - \{p\}$ if there exist an open set U and a continuum H in M such that $p \in U \subset H \subset M - N$. A continuum M is said to be *aposyndetic at a point* p if for each point q in $M - \{p\}$, M is aposyndetic at p with respect to q. If M is aposyndetic at each of its points, then M is said to be *aposyndetic*. The arcwise connected continua studied in [1] may be classified as follows:

A compact plane continuum M is said to be of *type* 1 if it is aposyndetic and contains a finite set of points F with the property that for each point x in $M - F$, there exist points y and z in F such that M is not aposyndetic at x with respect to $\{y, z\}$.

If M is semilocally connected at all except finitely many of its points and if at none of its points it is both aposyndetic and semilocally connected, then M is said to be of *type* 2.

If M is a continuum of type 1, then it is the sum of a finite number of cyclicly connected continua [1, Theorem 6]. Hence any two distinct points of M that are not separated in M by a point lie inside a simple closed curve contained in M. However, if M is of type 2, then M may contain two points that are not separated in M by a point and are not contained in a simple closed curve lying in M [1, Example 8].

A compact plane continuum H that does not separate the plane has another cyclic property. A point r in $H - \{p, q\}$ is said to *cut* p *from* q *in* H if each subcontinuum of H that contains $\{p, q\}$ also contains r. F. Burton Jones has shown that if p and q are distinct points of H and no point cuts p from q in H, then some simple closed curve in H contains p and q [3]. It is known that if a compact plane continuum M contains a point y such that for each point x in $M - \{y\}$, M is semilocally connected at x and M is not aposyndetic at x with respect to y, then M has Jones's cyclic property [1, Theorem 12]. Note that M is a continuum of type 2. It is the primary purpose of this paper to show that all continua of type 2 have Jones's cyclic property. To accomplish this, we first establish a theorem from which it follows that all continua of type 1 or 2 are hereditarily arcwise connected. Also, we give an example that rules out certain generalizations of this result.

Throughout this paper, S denotes the set of points of a simple closed surface (that is, a 2-sphere). For definitions of unfamiliar terms and phrases, see [4].

Definition. Let F be a finite set of points $\{y_1, y_2, \cdots, y_\alpha\}$ in a continuum M ($M \subset S$). For each i ($i = 1, 2, \cdots, \alpha$), let $\{V_n^i\}$ be a properly nested sequence of

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circular regions in S that are centered on y_i and converge to y_i. Suppose moreover that if $j = 1, 2, \cdots, \alpha$ and $j \neq i$, then $\mathrm{Cl}\, V_1^i \cap \mathrm{Cl}\, V_1^j = \emptyset$ ($\mathrm{Cl}\, V_1^i$ is the closure of V_1^i). For each positive integer j, let $V_j = \bigcup_{i=1}^{\alpha} V_j^i$. For a point x in $M - F$, let Y_j^x be the x-component of $M - V_j$. Let H_F^x be the limit superior of the sequence $\{Y_n^x\}$. Note that H_F^x is a subcontinuum of M and that it meets F.

THEOREM 1. *Suppose that* M *is a continuum in* S *and that* F *is a finite set of points in* M *such that for each point* x *in* $M - F$, M *is not aposyndetic at* x *with respect to* F *and* M *is aposyndetic at* x *with respect to each point of* $M - (F \cup \{x\})$. *Then* M *is hereditarily arcwise connected, and for each point* x *in* $M - F$, *the continuum* H_F^x *is hereditarily locally connected.*

Proof. Assume that there exists a point x in $M - F$ such that H_F^x contains a continuum H that is not locally connected. We shall prove that this assumption implies the existence of points p and w in $M - F$ such that M is not aposyndetic at p with respect to w; this will contradict the hypothesis of the theorem.

There exist two circular regions T and W in S with the properties that

(1) $T \supset \mathrm{Cl}\, W$,

(2) $F \cap \mathrm{Cl}\, T = \emptyset$, and

(3) there exists a sequence $\{H_n\}$ of disjoint continua in $H \cap (\mathrm{Cl}\, T - W)$ such that each continuum meets both $\mathrm{Bd}\, T$ (the boundary of T) and $\mathrm{Bd}\, W$, and such that the limit inferior Z of the sequence $\{H_n\}$ is a continuum

[4, Theorem 66 (proof), p. 124]. Let q be a point of Z that is not in $\mathrm{Bd}\, T \cup \mathrm{Bd}\, W$. Let $\{q_n\}$ be a sequence of points converging to q such that for each positive integer n, $q_n \in H_n \cap (T - \mathrm{Cl}\, W)$. Let $\{Q_n\}$ be a sequence of mutually disjoint circular regions in S, converging to q, such that for each positive integer n the region Q_n is centered on q_n and $\mathrm{Cl}\, Q_n$ is contained in $T - \mathrm{Cl}\, W$. For each positive integer n, there exists an integer i such that $Y_i^x \cap Q_n \neq \emptyset$. It follows that there exists a sequence $\{I_n\}$ of disjoint continua in $H_F^x \cap (\mathrm{Cl}\, T - W)$ such that to each n there corresponds an index i for which $I_n \cap Y_i^x$ is not empty, and such that the set $I = \liminf (I_1, I_2, \cdots)$ is a continuum that contains q and meets $\mathrm{Bd}(T - W)$. We can find a point p and two circular regions R and E, centered at q, such that

(1) $T - \mathrm{Cl}\, W \supset \mathrm{Cl}\, R \supset E$,

(2) $p \in (R - \mathrm{Cl}\, E) \cap I$,

(3) there exists a sequence $\{F_n\}$ of disjoint continua such that the set $\liminf (F_1, F_2, \cdots)$ is a continuum in H_F^x and contains p, and such that for each index n, F_n meets both $\mathrm{Bd}\, R$ and $\mathrm{Bd}\, E$, and there exists an integer j for which $I_j \cap (\mathrm{Cl}\, R - E) \supset F_n$.

Note that for each positive integer i, there exists an integer j such that $\bigcup_{n=1}^{i} F_n \subset Y_j^x$.

Concerning the sequence $\{F_n\}$, we may assume without loss of generality that for each positive integer n, there exist two arc-segments R_n and E_n such that

(1) $R_n \subset \mathrm{Bd}\, R$,

(2) $E_n \subset \mathrm{Bd}\, E$, and

(3) each arc-segment meets $F_1, F_2, F_3, \cdots$ only in F_{2n}, and it has one endpoint in F_{2n-1} and the other endpoint in F_{2n+1}.

Let $\{p_n\}$ be a sequence of points converging to p, and such that

$$p_n \in F_{2n} \cap (R - \mathrm{Cl}\, E)$$

for each n. The sequence $\{R_n\}$ converges to a point w_1 of $M \cap \mathrm{Bd}\, R$, and $\{E_n\}$ converges to a point w_2 of $M \cap \mathrm{Bd}\, E$.

Since M is aposyndetic at p with respect to each point of M - F, for n = 1 and 2, there exist subcontinua M_1 and M_2 of M and circular regions G_1 and G_2 in T such that $\mathrm{Cl}\, G_1 \cap \mathrm{Cl}\, G_2 = \emptyset$, and such that for n = 1 and 2, the set G_n contains w_n and meets only one component of Bd (R - E), the point p is in the interior of M_n relative to M, and $\mathrm{Cl}\, G_n \cap M_n = \emptyset$. Let G denote a circular region in S containing p, and such that

$$\mathrm{Cl}\, G \cap \mathrm{Cl}(G_1 \cup G_2) = \emptyset$$

and $G \cap M$ is contained in $M_1 \cap M_2$. Assume without loss of generality that for each positive integer i, $R_i \subset G_1$, $E_i \subset G_2$, and $p_i \in G$. Let j be a positive integer such that $\mathrm{Cl}\, V_j \cap \mathrm{Cl}\, T = \emptyset$ and the continua $F_1, F_2, \cdots, F_{2\alpha^2+1}$ all lie in Y^x_{j-1} (α is the cardinality of F). Let P_1 be a circular region in G, centered on p_1, and such that $\mathrm{Cl}\, P_1$ does not meet $F_1 \cup F_3 \cup R_1 \cup E_1$. Since M is not aposyndetic at p_1 with respect to F, the component of $M - V_j$ that contains p_1 is not open relative to M at p_1. Hence the boundary of P_1 contains an arc-segment S_1 whose endpoints a_1 and b_1 lie in different components of $M - V_j$, and such that $M \cap S_1 = \emptyset$.

There exists a simple closed curve C_1 that separates a_1 from b_1 in S and contains no point of $M - V_j$, and such that $C_1 \cap S_1$ is connected and C_1 intersects $\mathrm{Bd}\, E \cup \mathrm{Bd}\, R$ in only a finite number of points and crosses at each of these. In C_1, there exists an arc-segment T_1 that crosses S_1, contains no point of $M \cup \mathrm{Cl}\, V_j$, and has its endpoints in $\mathrm{Bd}\, V_j$. Let P_2 be a circular region in G, centered on p_2, and such that $\mathrm{Cl}\, P_2$ does not meet $F_3 \cup F_5 \cup R_2 \cup E_2 \cup T_1$. The component of $(M \cup S_1 \cup \mathrm{Cl}\, T_1) - V_j$ containing p_2 is not open relative to $M \cup S_1 \cup \mathrm{Cl}\, T_1$ at p_2. Hence the boundary of P_2 contains an arc-segment S_2 whose endpoints a_2 and b_2 lie in different components of $(M \cup S_1 \cup \mathrm{Cl}\, T_1) - V_j$ and whose intersection with M is empty.

There exists a simple closed curve C_2 that separates a_2 from b_2 in S and contains no point of $(M \cup S_1 \cup \mathrm{Cl}\, T_1) - V_j$, and such that $C_2 \cap S_2$ is connected and C_2 intersects $\mathrm{Bd}\, E \cup \mathrm{Bd}\, R$ in only a finite number of points and crosses at each of these. In C_2, there exists an arc-segment T_2 that crosses S_2, contains no point of $M \cup \mathrm{Cl}\, V_j$, and has its endpoints in $\mathrm{Bd}\, V_j$.

Continue this process. For each k ($k = 1, 2, \cdots, \alpha^2$), there exist a circular region P_k (centered on p_k) in G, arc-segments S_k and T_k, and a simple closed curve C_k such that

(1) $\mathrm{Cl}\, P_k$ does not meet $F_{2k-1} \cup F_{2k+1} \cup R_k \cup E_k \cup \bigcup_{i=1}^{k-1} T_i$,

(2) S_k has endpoints a_k and b_k in M and is contained in $(S - M) \cap \mathrm{Bd}\, P_k$,

(3) C_k separates a_k from b_k and contains no point of

$$\left(M \cup \bigcup_{i=1}^{k-1} (S_i \cup \mathrm{Cl}\, T_i)\right) - V_j,$$

(4) $C_k \cap S_k$ is connected and C_k intersects Bd E $\cup$ Bd R in only a finite number of points and crosses at each of these, and

(5) T_k is contained in C_k - Cl V_j, meets S_k, and has its endpoints in Bd V_j.

Statement C. *For each* k (k = 1, 2, $\cdots$, α^2), *no component of* $\text{Cl}(G_1 \cup G_2)$ *contains both a point of* T_k *that precedes and a point of* T_k *that follows* $T_k \cap S_k$ *with respect to the order of* T_k. Note that if Statement C were false, then for some k, the union of the arc-segment T_k and a component of $\text{Bd}(G_1 \cup G_2)$ would separate a_k from b_k in S, and this would contradict the existence of M_1 and M_2 [4, Theorem 32, p. 181].

Since there are α^2 arc-segments T_k, there exist components V and Y of Bd V_j and integers r and s such that the endpoints of T_r and T_s are contained in V $\cup$ Y. We now have two cases to consider:

I. One of T_r and T_s, say T_r, has both endpoints on the same component of Bd V_j.

II. Each of T_r and T_s has endpoints on different components of Bd V_j.

In case I, construct a simple closed curve J from T_r and an arc on Bd V_j; in case II, construct J from T_r, T_s, and two arcs on Bd V_j.

Note that E_r crosses J an even number of times, since its endpoints lie in the connected set $Y^x_{j-1} \subset S - J$. Hence $|E_r \cap T_r| = |E_r \cap T_s|$ (mod 2); that is, E_r intersects each of T_r and T_s an even number of times, or each of T_r and T_s an odd number of times.

E_r cannot cross T_r an even number of times, for the subarcs of T_r in the region bounded by $F_{2r-1} \cup F_{2r+1} \cup E_r \cup R_r$ have both endpoints on E_r or R_r, except for the one that meets S_r (otherwise, Statement C would be contradicted). This eliminates case I and reduces case II to the problem of showing that not both of $|E_r \cap T_r|$ and $|E_r \cap T_s|$ can be odd.

If $|E_r \cap T_s|$ is odd, then a component of $T_s - S_s$ intersects both G_1 and G_2, contrary to Statement C. It follows that for each point x of M the continuum H^x_F is hereditarily locally connected.

Now, to prove that M is hereditarily arcwise connected, let K be a subcontinuum of M. Assume that $K \cap F$ is void. Let p be a point of K. The set K is contained in the hereditarily locally connected continuum H^p_F, and it is therefore arcwise connected. Suppose that $K \cap F$ is not empty. For each point p of K - F, there exists a continuum L in $K \cap H^p_F$ that contains p and meets $K \cap F$. If $K \cap F$ consists of one point, then K is arcwise connected. Assume that $K \cap F$ contains more than one point. Let A and B be nonempty disjoint subsets of F such that $A \cup B = K \cap F$. Since K is a continuum in M, there exists a point p in K - F such that $K \cap H^p_F$ contains a continuum that meets both A and B. Therefore any two distinct points of $K \cap F$ are the endpoints of an arc contained in K. It follows that K is arcwise connected. Hence M is hereditarily arcwise connected.

COROLLARY. *If* M *is a compact plane continuum of type* 1 *or type* 2, *then* M *is hereditarily arcwise connected.*

Definitions. For a point y of a continuum M, Jones defines L_y to be the subcontinuum of M consisting of y and all points x in M - $\{y\}$ such that M is not aposyndetic at x with respect to y [2]. Let x and y be distinct points of a compact metric continuum M such that M is not aposyndetic at x with respect to y. Let

$\{V_n\}$ be a monotone descending sequence of circular open subsets of M that are centered on y and converge to y. For each positive integer n, let Y_n^x be the x-component of $L_y - V_n$. Define L_y^x to be the limit superior of $\{Y_n^x\}$. The set L_y^x is a subcontinuum of M that contains x and y. The continuum M is not aposyndetic at any point of $L_y^x - \{y\}$ with respect to y.

THEOREM 2. *Suppose that* M *is a compact plane continuum that is semilocally connected at all except finitely many of its points, and is at none of its points both aposyndetic and semilocally connected. If* p *and* q *are distinct points of* M *and no point cuts* p *from* q *in* M, *then there exists a simple closed curve in* M *that contains* p *and* q.

Proof. Assume that no point cuts the point p from the point q in M. Since M is hereditarily arcwise connected, it is sufficient to show that there exist simple closed curves J_p and J_q with the property that $p \in J_p$, $q \in J_q$, and either $J_p \cap J_q \neq \emptyset$ or there exists an arc-segment A in $M - (J_p \cup J_q)$ such that one endpoint of A lies in $J_p - \{p\}$ and the other endpoint of A lies in $J_q - \{q\}$.

Case 1. Assume there exist points x and y in M such that $\{p, q\} \subset L_y^x$. The continuum L_y^x is locally connected [1, Theorem 8]. If no point separates p from q in L_y^x, then some simple closed curve in L_y^x contains p and q. Suppose that some point separates p from q in L_y^x. There are only finitely many points that separate L_y^x [1, Theorem 10]. Hence there exists a point r of L_y^x that separates p from q in L_y^x, and such that no other point separates p from r in L_y^x. There exists a point s of L_y^x that separates p from q in L_y^x, and such that no other point separates q from s in L_y^x. There exist simple closed curves J_p and J_q in L_y^x that contain $\{p, r\}$ and $\{q, s\}$, respectively. If $r = s$, the conclusion follows immediately. Assume that $r \neq s$. Since both r and s separate p from q in L_y^x, there exists an arc in L_y^x from J_p to J_q that does not meet $\{p, q\}$.

Case 2. Suppose there exists a point y in M such that $\{p, q\} \subset L_y$, $p \neq y$, and $q \notin L_y^p$. By the preceding argument, there exists a simple closed curve J_p containing p in L_y^p such that either $y \in J_p$ or else the set $L_y^p - \{p\}$ contains an arc B from J_p to y. Also, there exists a simple closed curve J_q in L_y^q such that either $y \in J_q$ or else there is an arc C in $L_y^q - \{q\}$ from J_q to y. If $J_p \cap J_q = \emptyset$, there exists an arc in $L_y - \{p, q\}$ from J_p to J_q.

Case 3. Suppose there exists no point y in M such that $\{p, q\} \subset L_y$. It follows that there exists a finite set of points $\{y_1, y_2, \cdots, y_n\}$ such that

(1) $\bigcup_{i=1}^n L_{y_i}$ is a continuum,

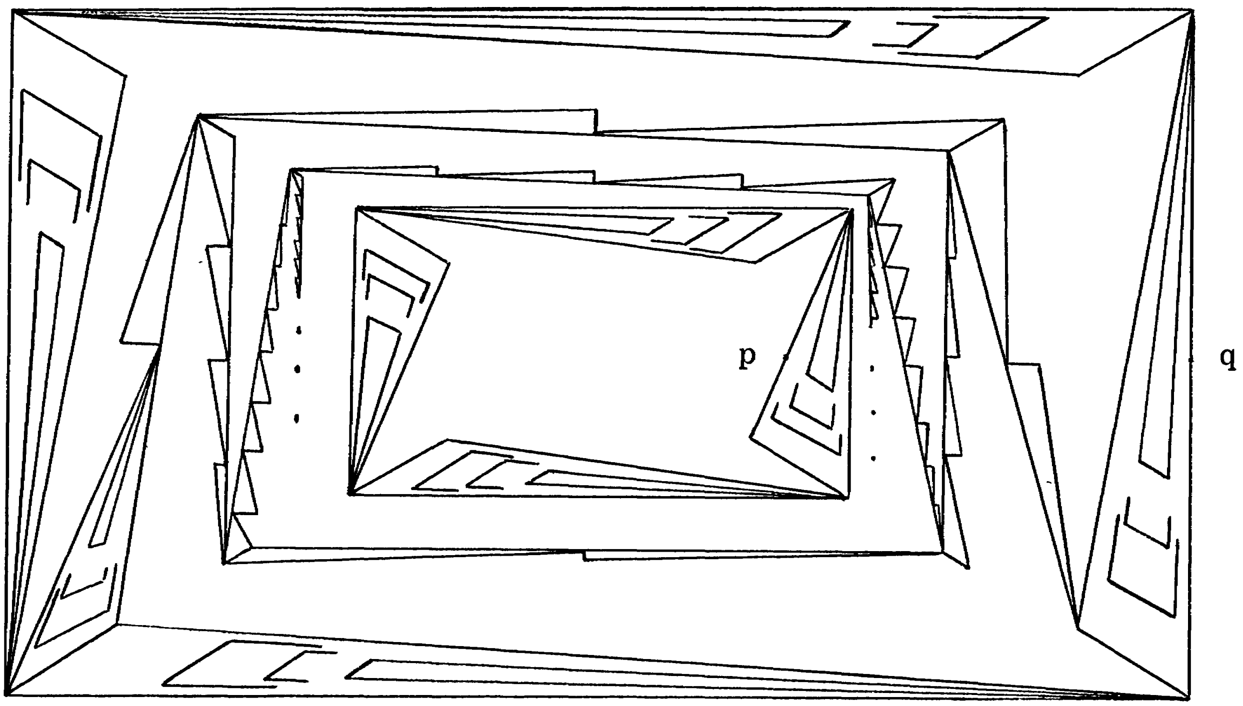
(2) $p \in L_{y_1} - \bigcup_{i=2}^n L_{y_i}$, and

(3) $q \in L_{y_n} - \bigcup_{i=1}^{n-1} L_{y_i}$.

Assume first that $L_{y_1} \cap L_{y_n}$ contains a point w. Note that $p \neq w$ and $q \neq w$. In L_{y_1}, there exists a simple closed curve J_p, containing p, such that either $w \in J_p$ or there is an arc B in $L_{y_1} - \{p\}$ from J_p to w. There exists a simple closed curve J_q in L_{y_n} that contains q, and such that either $w \in J_q$ or there exists an arc

C in $L_{y_n} - \{q\}$ from J_q to w. The conclusion follows immediately. If $L_{y_1} \cap L_{y_n} = \emptyset$, then there exists an arc in $\bigcup_{i=2}^{n-1} L_{y_i}$ that has one endpoint in L_{y_1} and the other endpoint in L_{y_n}. Hence there exist simple closed curves J_p in L_{y_1} and J_q in L_{y_n}, containing p and q, respectively, and there exists an arc in $M - \{p, q\}$ from J_p to J_q.

Example. Suppose that M is a compact plane continuum that is not both aposyndetic and semilocally connected at any of its points. If M is semilocally connected at all but countably many of its points, then M may fail to have Jones's cyclic property. To see this, consider the compact plane continuum M in the figure. M is the union of infinitely many Cantor suspensions, each suspension having its endpoints identified. (A Cantor suspension is a continuum that is the upper-semicontinuous decomposition of the topological product of the unit interval $[0, 1]$ and the Cantor discontinuum C in which the sets $0 \times C$ and $1 \times C$ are points. These sets are called endpoints of the Cantor suspension.) No point cuts p from q in M, and M does not contain an arc from p to q. Hence M does not have the cyclic property. The suspensions in M are countable, and M is semilocally connected at each point that is not an endpoint of some suspension. M is totally nonaposyndetic.



REFERENCES

1. C. L. Hagopian, *Concerning arcwise connectedness and the existence of simple closed curves in plane continua.* Trans. Amer. Math. Soc. 147 (1970), 389-402.

2. F. B. Jones, *Concerning non-aposyndetic continua.* Amer. J. Math. 70 (1948), 403-413.

3. ———, *The cyclic connectivity of plane continua.* Pacific J. Math. 11 (1961), 1013-1016.

4. R. L. Moore, *Foundations of point set theory*. Revised edition. Amer. Math. Soc. Colloquium Publications, Vol. XIII. Amer. Math. Soc., Providence, R.I., 1962.

5. G. T. Whyburn, *Cyclicly connected continuous curves*. Proc. Nat. Acad. Sci. U.S.A. 13 (1927), 31-38.

California Institute of Technology
Pasadena, California 91109
and
Sacramento State College
Sacramento, California 95819