THE RELATIVE GROWTH OF SUBORDINATE FUNCTIONS

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1. INTRODUCTION

Suppose the functions f and F are regular in the unit disk K and vanish at the origin. The function f is said to be subordinate to F in K (in symbols: $f \prec F$) if there exists a function ω regular in K with the properties that $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in K)$, and $f(z) \equiv F(\omega(z))$. In all sufficiently small disks $K_r = \{z: |z| < r\}$, functionals of r and f are in general dominated by corresponding functionals of r and F, whenever $f \prec F$. Many authors have studied the problem of determining the largest disk where such a domination takes place. For example, G. M. Golusin [4] proved the following result. Let a(r) and A(r) denote the areas of the Riemann surfaces $f(K_r)$ and $F(K_r)$, respectively. Then

$$a(r) \leq A(r) \qquad (0 \leq r \leq 1/\sqrt{2}),$$

provided $f \prec F$. E. Reich was the first to investigate a more general problem. He obtained estimates of the ratio a(r)/A(r) in the whole unit disk under the assumption that $f \prec F$, and he proved the inequality [7]

$$a(r)/A(r) \leq mr^{2m-2} \qquad \left(\frac{m-1}{m} \leq r^2 \leq \frac{m}{m+1};\ m = 1, 2, \cdots\right),$$

which implies Golusin's result in the case where $m = 1$.

In this paper, we study the least upper bound of another ratio. The authors thank Professor J. G. Krzyż for suggesting this problem.

Let A_n $(n = 1, 2, \cdots)$ denote the class of functions f regular in K such that

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (a_n \geq 0).$$

Let S denote the class of functions regular and univalent in K, subject to the usual normalizations. Suppose S_0 is some fixed subclass of S, and suppose that for each η $(|\eta| < 1)$, the function $\eta^{-1} f(\eta z)$ belongs to S_0 whenever $f \in S_0$. Define

$$\kappa(r, n, S_0) = \sup\{|f(z)/F(z)|: f \in A_n,\ F \in S_0,\ f \prec F,\ |z| = r\}$$

(n is a positive integer, and $0 < r < 1$). We are able to determine $\kappa(r, n, S^*)$ and $\kappa(r, n, S_c)$, where S^* denotes the class of functions starlike with respect to the origin and S_c denotes the class of convex functions.

Let B_n $(n = 1, 2, \cdots)$ denote the class of functions ω regular in K and satisfying the conditions

$$\omega(z) = \alpha_n z^n + \alpha_{n+1} z^{n+1} + \cdots \qquad (\alpha_n \geq 0)$$

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and

$$|\omega(z)| < 1 \qquad (z \in K).$$

Arguments similar to those used in [5] show that for a fixed $z_1 \in K$, the set of all possible values of $\omega(z_1)$ ($\omega \in B_n$) constitutes the closed domain $H_n(z_1)$ (generalized Rogosinski domain) whose boundary consists of three arcs:

(1.1) $z = z_0(\theta) = |z_1|^{n+1} e^{i\theta} \qquad \left(\arg z_1^n + \frac{1}{2}\pi \leq \theta \leq \arg z_1^n + \frac{3}{2}\pi\right),$

(1.2) $z = z_1(\alpha) = z_1^n(\alpha + i|z_1|)/(1 + i\alpha|z_1|) \qquad (0 \leq \alpha \leq 1),$

(1.3) $z = z_2(\alpha) = z_1^n(\alpha - i|z_1|)/(1 - i\alpha|z_1|) \qquad (0 \leq \alpha \leq 1).$

We write $Q_n(z_1, S_0) = \{u: u = F(z_2)/F(z_1)\}$, where z_1 is a fixed point of K, z_2 ranges over $H_n(z_1)$, and F ranges over S_0. Under our assumptions on S_0, the set $Q_1(z_1, S_0)$ has the following properties (see [2]):

(1.4) $$Q_1(z_1, S_0) = Q_1(|z_1|, S_0),$$

(1.5) $$\text{if } 0 < r_1 < r_2 < 1, \text{ then } Q_1(r_1, S_0) \subset Q_1(r_2, S_0).$$

2. MAIN RESULTS

Let

$$\Omega_n(z_1, S_0) = \{w: w = f(z_1)/F(z_1),\ f \in A_n,\ F \in S_0,\ f \prec F\},$$

where z_1 is a fixed point of K.

THEOREM 1. $\Omega_n(z_1, S_0) = Q_n(z_1, S_0)$.

Proof. Suppose $u \in \Omega_n(z_1, S_0)$. This means that there exist $f \in A_n$ and $F \in S_0$ such that $f \prec F$ and $u = f(z_1)/F(z_1)$. The condition $f \prec F$ implies that there exists $\omega \in B_n$ such that $f(z) \equiv F(\omega(z))$, and hence $f(z_1) = F(\omega(z_1))$. If $z_2 = \omega(z_1)$, then $z_2 \in H_n(z_1)$ [5]. We now have the relations

$$u = f(z_1)/F(z_1) = F(\omega(z_1))/F(z_1) = F(z_2)/F(z_1),$$

and thus $u \in Q_n(z_1, S_0)$.

Suppose now $q \in Q_n(z_1, S_0)$. Then $q = F(z_2)/F(z_1)$, where $F \in S_0$ and $z_2 \in H_n(z_1)$. By [5], there exists $\omega \in B_n$ such that $z_2 = \omega(z_1)$. Consequently,

$$q = F(\omega(z_1))/F(z_1) = f(z_1)/F(z_1),$$

where $f = F \circ \omega \in A_n$. Hence $q \in \Omega_n(z_1, S_0)$.

COROLLARY 1. *If* $z \in \overline{K_r}$, $f \in A_n$, $F \in S_0$, *and* $f \prec F$, *then*

$$\sup |f(z)/F(z)| = \sup \{|w|: w \in \Omega_n(z, S_0)\}.$$

By Corollary 1, we have the relations

$$\kappa(r, n, S_0) = \sup\{|w| : w \in Q_n(z, S_0)\} = \sup\{|w| : w \in \Omega_n(z, S_0)\}.$$

Let $S^*_{1/2}$ denote the class of all functions $F \in S^*$ that satisfy the inequality

$$\Re\,(zF'(z)/F(z)) > 1/2 \qquad (z \in K).$$

THEOREM 2. *Suppose* $S_0 = S^*_{1/2}$, *and let* $0 < r < 1$. *Then*

$$\kappa(r, 1, S^*_{1/2}) = \max\left\{1, \frac{r}{1-r}\right\}, \tag{2.1}$$

$$\kappa(r, n, S^*_{1/2}) = r^{n-1}\frac{1+r}{1-r^n} \qquad (n \geq 2). \tag{2.2}$$

It is well known [6] that $S_c \subset S^*_{1/2}$. Hence $|f(z)/F(z)| \leq \kappa(r, n, S^*_{1/2})$, provided $F \in S_c$, $f \in A_n$, and $f \prec F$. On the other hand, if $F(z) = z/(1+z)$ and $f(z) \equiv F(z)$, then $|f(z)/F(z)| \equiv 1$, and for the same F and $f(z) = F(-z^2)$ we have the relation

$$\sup_{|z|=r} |f(z)/F(z)| = r/(1-r).$$

Thus we have established the following result.

COROLLARY 2. $\kappa(r, 1, S_c) = \max\{1, r/(1-r)\}$ $(0 < r < 1)$.

For $n \geq 2$, let

$$F(z) = z\left(1 - z\exp\frac{in\pi}{n-1}\right)^{-1} \qquad \text{and} \qquad f(z) = F(z^n).$$

Then $F \in S_c$, $f \in A_n$, and for $z_0 = r\exp\frac{-i\pi}{n-1}$, we obtain the relation

$$|f(z)/F(z)| = r^{n-1}(1+r)/(1-r^n).$$

COROLLARY 3. $\kappa(r, n, S_c) = r^{n-1}\dfrac{1+r}{1-r^n}$ $(n \geq 2,\ 0 < r < 1)$.

THEOREM 3. *Suppose* $S_0 = S^*$, *and let* $0 < r < 1$. *Then*

$$\kappa(r, 1, S^*) = \max\{1, r(1-r)^{-2}\} \quad \textit{and} \quad \kappa(r, n, S^*) = r^{n-1}\left(\frac{1+r}{1-r^n}\right)^2 \quad (n \geq 2).$$

3. PROOFS OF THEOREMS 2 AND 3

Proof of Theorem 2. Suppose $n = 1$. In [3], we have proved that $\kappa(r, 1, S^*_{1/2}) = 1$ $(0 < r \leq 1/2)$; hence we may assume that $1/2 < r < 1$. If $F \in S^*_{1/2}$ and $z_1, z_2 \in K$ $(z_1 \neq 0)$, then the point $u = F(z_2)/F(z_1)$ lies inside the circle

$$w(\theta) = \frac{z_2}{z_1}(1 - z_1 e^{-i\theta})(1 - z_2 e^{-i\theta})^{-1} \qquad (-\pi \leq \theta \leq \pi) \tag{3.1}$$

(see [8]). The center s and radius R of this circle are

$$s = \frac{z_2/z_1 - |z_2|^2}{1 - |z_2|^2}, \tag{3.21}$$

$$R = \left|\frac{z_2}{z_1}\right| \cdot \frac{|z_2 - z_1|}{1 - |z_2|^2}. \tag{3.22}$$

The boundary $\partial H_1(r)$ of $H_1(r)$ consists of three circular arcs with equations

$$z = z_0(t) = r^2 e^{it} \qquad \left(\frac{1}{2}\pi \leq t \leq \frac{3}{2}\pi\right), \tag{3.3}$$

$$z = z_1(\alpha) = r(\alpha + ir)/(1 + i\alpha r) \qquad (0 \leq \alpha \leq 1), \tag{3.4}$$

$$z = z_2(\alpha) = r(\alpha - ir)/(1 - i\alpha r) \qquad (0 \leq \alpha \leq 1). \tag{3.5}$$

For fixed θ, for $z_1 = r$, and for $z_2 \in H_1(r)$, the right-hand side of (3.1) is an analytic function of z_2; hence the boundary points of $Q_1(r, S^*_{1/2})$ correspond to points $z_2 \in \partial H_1(r)$. In order to establish relation (2.1), we must verify that the inequality

$$h(z_2, \theta) = \frac{|z_2|}{r}\left|\frac{1 - re^{-i\theta}}{1 - z_2 e^{-i\theta}}\right| \leq \frac{r}{1 - r} \qquad (z_2 \in \partial H_1(r), -\pi \leq \theta \leq \pi)$$

holds and that equality is attained for each $r \in (1/2, 1)$.

Recall that the boundary of $H_1(r)$ consists of three circular arcs with equations (3.3), (3.4), and (3.5). If z_2 lies on the arc (3.3), then

$$h(z_2, \theta) = r\left|\frac{1 - re^{-i\theta}}{1 - r^2 e^{-i(\theta - t)}}\right| \leq r\,\frac{1 + r}{1 - r^2} = \frac{r}{1 - r},$$

and equality is attained for $\theta = t = \pi$. We proceed to verify that the inequality $h(z_2, \theta) \leq r/(1 - r)$ holds on the arcs (3.4) and (3.5). By (3.1), (3.21), and (3.22), we need to verify the inequality

$$|s| + R \leq r/(1 - r) \qquad (1/2 < r < 1).$$

Using (3.21), (3.22), and (3.4) or (3.5), we find that the inequality above becomes

$$\frac{(\alpha^2 + r^2)^{1/2}}{1 - r^4}[((1 - \alpha r^2)^2 + r^2(\alpha - r^2)^2)^{1/2} + r(1 - \alpha)(1 + r^2)^{1/2}] \leq \frac{r}{1 - r} \qquad (1/2 < r < 1,\ 0 < \alpha < 1). \tag{3.6}$$

In order to verify (3.6), we use an elementary but tedious argument. We first multiply both sides of (3.6) by $(1 - r^4)(1 + r^2)^{-1/2}$, and then we subtract the term $r(1 - \alpha)(\alpha^2 + r^2)^{1/2}$ from both sides. Having squared both sides and rearranged terms, we find the equivalent inequality

$$\alpha^2(1 - r)^2(1 + r) + 2r^2(1 - \alpha)\sqrt{(\alpha^2 + r^2)(1 + r^2)} \leq 2r^3(1 + r). \tag{3.7}$$

We define the auxiliary functions

$$\phi(\alpha, r) = \alpha^2(1-r)^2(1+r) + 2r^2(1-\alpha)\sqrt{(\alpha^2+r^2)(1+r^2)},$$

$$\psi(r) = 2r^3\sqrt{1+r^2} + (3-\sqrt{5})/8,$$

$$\chi(r) = 2r^3(1+r),$$

and we shall show that $\phi(\alpha, r) \leq \psi(r) \leq \chi(r)$ $(1/2 < r < 1,\ 0 < \alpha < 1)$.

It is easy to verify that the expression

$$k(1-\alpha)\sqrt{\alpha^2+c^2} \qquad (k > 0,\ 1/2 \leq c \leq 1) \tag{3.8}$$

decreases as α increases $(0 < \alpha < 1)$. A straightforward computation shows that

$$\phi(\alpha, 1/2) \leq 3/8 = \psi(1/2) \qquad (0 < \alpha < 1). \tag{3.9}$$

Moreover, $\phi_r'(\alpha, r) \leq \psi'(r)$ $(1/2 < r < 1)$, where $\phi_r'(\alpha, r) = \partial\phi(\alpha, r)/\partial r$. In fact,

$$\phi_r'(\alpha, r) = \alpha^2(3r^2 - 2r - 1) + 4r(1-\alpha)((\alpha^2+r^2)(1+r^2))^{1/2}$$
$$+ 2r^3(1-\alpha)[(\alpha^2+r^2)^{-1} + (1+r^2)^{-1}]((\alpha^2+r^2)(1+r^2))^{1/2}.$$

We see that $\phi_r'(\alpha, r)$ has the form (3.8), and therefore

$$\phi_r'(\alpha, r) \leq \phi_r'(0, r) = \psi'(r).$$

The inequality above, together with (3.9), implies that $\phi(\alpha, r) \leq \psi(r)$. The inequality $\psi(r) \leq \chi(r)$ is obvious. We have verified inequality (3.7), and therefore (3.6), and Theorem 2 is established for the case $n = 1$.

Now suppose $n \geq 2$. The boundary $\partial H_n(z_1)$ of the generalized Rogosinski domain $H_n(z_1)$ is given by equations (1.1), (1.2), and (1.3). Setting $z_1 = re^{it}$, we obtain from (1.1) and (3.1) the relation

$$\max_{t,\theta} |w(t, \theta)| = |w(0, \pi)| = r^n \frac{1+r}{1-r^{n+1}}.$$

Moreover, using (3.1) together with (1.2) or (1.3), we find that

$$\max_{\alpha,t,\theta} |w(\alpha, t, \theta)| = \left|w\left(1, -\frac{1}{n-1}\pi, -\frac{n}{n-1}\pi\right)\right| = r^{n-1}\frac{1+r}{1-r^n}.$$

It follows that for $n \geq 2$,

$$\kappa(r, n, S^*_{1/2}) = r^{n-1}\frac{1+r}{1-r^n} \qquad (0 < r < 1),$$

and Theorem 2 is proved.

Proof of Theorem 3. Suppose first that $n = 1$. If $0 < r < (3-\sqrt{5})/2$, then $\kappa(r, 1, S^*) = 1$ [4]. Thus we may assume that $(3-\sqrt{5})/2 < r < 1$. It is well known [8] that the domain of variability of the point $u = [F(z_2)/F(z_1)]^{1/2}$, where z_1 $(z_1 \neq 0)$ and z_2 are fixed points in K and where F ranges over S^*, is the closed disk with boundary

$$w(\theta) = \left(\frac{z_2}{z_1}\right)^{1/2} \frac{1 - z_1 e^{-i\theta}}{1 - z_2 e^{-i\theta}} \quad (-\pi \leq \theta \leq \pi). \tag{3.10}$$

The center S and radius R of this disk are

$$S = (q - |z_2|^2 q^{-1})/(1 - |z_2|^2), \tag{3.111}$$

$$R = |z_2|\,|q - q^{-1}|/(1 - |z_2|^2), \tag{3.112}$$

where $q = (z_2/z_1)^{1/2}$. An argument similar to the one used in the proof of Theorem 2 leads to the inequality

$$\frac{[(\alpha^2 + r^2)(1 + \alpha^2 r^2)]^{1/2}}{(1 - r^4)^2} [((1 - \alpha r^2)^2 + r^2(\alpha - r^2)^2)^{1/2} + r(1 - \alpha)\sqrt{1 + r^2}\,]^2 \leq \frac{r}{(1 - r)^2}, \tag{3.12}$$

where $0 < \alpha < 1$ and $(3 - \sqrt{5})/2 < r < 1$. For $0 < \alpha < 1$ and $0 < r < \sqrt{2} - 1$, the left-hand side of (3.12) does not exceed 1 [2]. Hence it suffices to verify (3.12) for $0 < \alpha < 1$ and $2/5 \leq r < 1$. One can show that (3.12) is equivalent to the inequality

$$\begin{aligned} &((\alpha^2 + r^2)(1 + \alpha^2 r^2))^{1/2}(1 - r^2)(1 + r) \\ &+ 2r(1 - \alpha)((\alpha^2 + r^2)(1 + \alpha^2 r^2))^{1/4}(r(1 + r^2))^{1/2} \leq r(1 + 1)(1 + r^2). \end{aligned} \tag{3.13}$$

Calculations similar to those in the proof of Theorem 2, this time with the auxiliary functions

$$\begin{aligned} \phi(\alpha, r) &= ((\alpha^2 + r^2)(1 + \alpha^2 r^2))^{1/2}(1 - r)^2(1 + r) \\ &\quad + 2r(1 - \alpha)((\alpha^2 + r^2)(1 + \alpha^2 r^2))^{1/4}(r(1 + r^2))^{1/2}, \\ \psi(r) &= 2r^2\sqrt{1 + r^2} + \frac{8}{125}\sqrt{29} - \frac{406}{625}, \quad \text{and} \\ \chi(r) &= r(1 + r)(1 + r^2), \end{aligned}$$

show that $\phi(\alpha, r) \leq \psi(r) \leq \chi(r)$. These inequalities establish (3.13), and therefore (3.12). It follows that $\kappa(r, 1, S^*) = \max\{1, r(1 - r)^{-2}\}$.

The assertion about $\kappa(r, n, S^*)$ $(n \geq 2)$ can be proved with arguments similar to those we used to establish the bound $\kappa(r, n, S^*_{1/2})$.

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