UNIFORM CONVERGENCE OF FOURIER SERIES ON GROUPS, I

C. W. Onneweer and Daniel Waterman

In 1940, R. Salem [6] proved that the Fourier series of a continuous periodic function converges uniformly if $\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} Q_n(x) = 0$ uniformly in x, where

$$T_n(x) = \sum_{p=0}^{(n-1)/2} (p+1)^{-1}[f(x+2p\pi/n) - f(x+(2p+1)\pi/n)],$$

and where $Q_n(x)$ is obtained from $T_n(x)$ by changing π into $-\pi$. In a recent paper [5], C. W. Onneweer proved a similar theorem for Walsh-Fourier series. In this paper, we extend this result to continuous functions defined on certain compact, 0-dimensional, metrizable, abelian groups. Such groups and their character groups were first studied by N. Ja. Vilenkin [7]. The significance of our result is evidenced primarily by its corollaries, as was the case with Salem's original theorem.

1. THE GROUPS G AND X

Let G be a compact, 0-dimensional, metrizable, abelian group, and let X be its character group. Then X is a discrete, countable, abelian torsion group [4, Theorems 24.15 and 24.26]. Vilenkin [7, Sections 1.1 and 1.2] established the existence of an increasing sequence of finite subgroups $\{X_n\}$ of X such that

(i) $X_0 = \{\chi_0\}$, where χ_0 is the identity character on G,

(ii) each X_n/X_{n-1} is of prime order p_n, and

(iii) $X = \bigcup_{n=0}^{\infty} X_n$.

Moreover, the subgroups X_n can be chosen so that there exists a sequence $\{\phi_n\}$ of elements of X satisfying the conditions

$$\text{(i) } \phi_n \in X_{n+1} \setminus X_n \quad \text{and} \quad \text{(ii) } \phi_n^{p_{n+1}} \in X_n.$$

Using these ϕ_n, we can enumerate the elements of X as follows. Let $m_0 = 1$ and $m_n = \prod_{i=1}^{n} p_i$. Each natural number k can be represented uniquely as $k = \sum_{i=0}^{s} a_i m_i$, with $0 \leq a_i < p_{i+1}$ for $0 \leq i \leq s$; we define χ_k by the formula $\chi_k = \phi_0^{a_0} \cdot \ldots \cdot \phi_s^{a_s}$. Then $X_n = \{\chi_i \mid 0 \leq i < m_n\}$.

Next, let G_n be the annihilator of X_n, that is, let

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$$G_n = \{x \in G \mid \chi_k(x) = 1 \text{ for } 0 \leq k < m_n\}.$$

Then, obviously, $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ and $\bigcap_{n=0}^{\infty} G_n = \{0\}$, and it is easy to show that the G_n form a basis for the neighborhoods of zero in G. In [7, Section 3.2] Vilenkin proved that for each n there is an $x_n \in G_n \setminus G_{n+1}$ such that $\chi_{m_n}(x_n) = e^{2\pi i/p_{n+1}}$. He also observed that each $x \in G$ has a unique representation $x = \sum_{i=0}^{\infty} b_i x_i$, with $0 \leq b_i < p_{i+1}$. This representation of the elements of G enables us to order them by means of the lexicographic ordering of the corresponding sequences $\{b_n\}$. Furthermore,

$$G_n = \{x \in G \mid x = \sum_{i=0}^{\infty} b_i x_i \text{ with } b_0 = \cdots = b_{n-1} = 0\}.$$

Consequently, each coset of G_n in G has a representation of the form $z + G_n$, where $z = \sum_{i-0}^{n-1} b_i x_i$ for some choice of the b_i with $0 \leq b_i < p_{i+1}$. We shall denote these z, ordered lexicographically, by $\{z_\alpha^{(n)}\}$ $(0 \leq \alpha < m_n)$.

Remark 1. The choice of the $\phi_n \in X$ and of the $x_n \in G$ is not uniquely determined by the groups X and G. In the following, we assume that a particular choice has been made.

Remark 2. The standard examples of groups G and X as described by Vilenkin are

(a) $G = \prod_{n=1}^{\infty} (Z(2))_n$; here X is the group of Walsh functions as described by N. J. Fine [3];

(b) $G = \prod_{n=1}^{\infty} (Z(p))_n$; here X is the group of generalized Walsh functions; see [2].

2. FOURIER SERIES OF FUNCTIONS ON G, AND DIRICHLET KERNELS

Let dx denote the normalized Haar measure on G. If $f \in L_1(G)$, then the Fourier series of f is the series

$$S[f](x) = \sum_{i=0}^{\infty} c_i \chi_i(x), \qquad \text{where } c_i = \int_G f(t)\overline{\chi_i(t)}\,dt.$$

For the partial sums of $S[f]$ we have the formula

$$S_n(x; f) = \sum_{i=0}^{n-1} c_i \chi_i(x) = \int_G f(x-t) D_n(t)\,dt,$$

where $D_n(t) = \sum_{i=0}^{n-1} \chi_i(t)$. $D_n(t)$ is called the Dirichlet kernel of order n. We shall now state a number of properties of the Dirichlet kernels.

LEMMA 1. *For each* n,

$$D_{m_n}(x) = \begin{cases} 0 & (x \notin G_n), \\ m_n & (x \in G_n). \end{cases}$$

See [7, Section 2.2] for a proof.

LEMMA 2. *For each* n, $\int_G D_n(t)\,dt = 1$.

Proof. This follows from the fact that $\int_G \chi_k(t)\,dt = 0$ for $k > 0$.

LEMMA 3. *If* $m_k < n \le m_{k+1}$ *and* $n = a_k m_k + n'$, *with* $0 < a_k \le p_{k+1}$ *and* $0 \le n' < m_k$, *then*

$$D_n(x) = \frac{1 - \chi_{m_k}^{a_k}(x)}{1 - \chi_{m_k}(x)} D_{m_k}(x) + \chi_{m_k}^{a_k}(x)\, D_{n'}(x).$$

Proof.

$$D_n(x) = \sum_{i=0}^{a_k m_k - 1} \chi_i(x) + \sum_{i=a_k m_k}^{n-1} \chi_i(x) = \sum_{j=0}^{a_k-1} \sum_{i=0}^{m_k-1} \chi_{jm_k+i}(x) + \sum_{i=0}^{n'-1} \chi_{a_k m_k+i}(x)$$

$$= \sum_{j=0}^{a_k-1} \chi_{m_k}^{j}(x) \sum_{i=0}^{m_k-1} \chi_i(x) + \chi_{m_k}^{a_k}(x) \sum_{i=0}^{n'-1} \chi_i(x)$$

$$= \frac{1 - \chi_{m_k}^{a_k}(x)}{1 - \chi_{m_k}(x)} D_{m_k}(x) + \chi_{m_k}^{a_k}(x)\, D_{n'}(x).$$

LEMMA 4. *If* $x \in G \setminus G_n$, *then* $|D_k(x)| \le m_n$ *for all* k.

See [7, Section 3.61] for a proof.

Before stating Lemma 5, which is a generalization of Lemma 1 in [3], we need a definition.

Definition 1. G has property (P) if $\sup_i (p_i) = p < \infty$.

LEMMA 5. *If* G *has property* (P), *then* $|D_k(z_\alpha^{(n)})| < (p+1)m_n/\alpha$ *for all* k, n, *and* α $(0 < \alpha < m_n)$.

Proof. For each $z_\alpha^{(n)}$, there exists an ℓ $(0 \le \ell < n)$ such that $z_\alpha^{(n)} \in G_\ell \setminus G_{\ell+1}$. Consequently,

$$z_\alpha^{(n)} = \sum_{i=\ell}^{n-1} b_i x_i, \qquad \text{with } b_\ell \ne 0 \text{ and } 0 \le b_i < p_{i+1}.$$

Also,

$$\alpha = b_\ell p_{\ell+2} \cdot \cdots \cdot p_n + b_{\ell+1} p_{\ell+3} \cdot \cdots \cdot p_n + \cdots + b_{n-2} p_n + b_{n-1}.$$

Therefore,

$$m_{\ell+1} \alpha/m_n = b_\ell + \frac{b_{\ell+1}}{p_{\ell+2}} + \cdots + \frac{b_{n-2}}{p_{\ell+2} \cdot \cdots \cdot p_{n-1}} + \frac{b_{n-1}}{p_{\ell+2} \cdot \cdots \cdot p_n} < b_\ell + 2 \leq p+1.$$

Hence, Lemma 4 implies that

$$|D_k(z_\alpha^{(n)})| \leq m_{\ell+1} < (p+1) m_n/\alpha.$$

3. THE MAIN THEOREM

In this section we prove our main result, which is similar to Salem's theorem for trigonometric Fourier series [6, Chapter VI]. For another proof of Salem's result, see [1, Chapter IV, Section 5]. The special case of Theorem 1, which we obtain when $G = \prod_{n=1}^{\infty} (Z(2))_n$, was proved by Onneweer in [5].

Definition 2. If f is a function on G and $H \subset G$, then

$$\operatorname{osc}(f, H) = \sup\{|f(x_1) - f(x_2)| \,|\, x_1, x_2 \in H\}.$$

Definition 3. If f is a function on G, then for each n

$$\theta_n(f) = \sup\{|f(x_1) - f(x_2)| \,|\, x_1, x_2 \in G \text{ and } x_1 - x_2 \in G_n\}.$$

We observe that f is continuous on G if and only if $\lim_{n\to\infty} \theta_n(f) = 0$.

THEOREM 1. *Let* G *satisfy condition* (P). *Let* f *be a continuous function on* G *such that*

$$\lim_{k\to\infty} \sum_{\alpha=1}^{m_k-1} \frac{1}{\alpha} \left| \sum_{j=0}^{p_{k+1}-1} f(x - z_\alpha^{(k)} - jx_k) e^{2\pi i j a_k/p_{k+1}} \right| = 0$$

uniformly in $x \in G$ *and* $a_k \in \{1, 2, \cdots, p_{k+1} - 1\}$. *Then the Fourier series of* f *converges uniformly on* G.

Proof. Let $n = \sum_{i=0}^{k} a_i m_i$, with $a_k \neq 0$ and $0 \leq a_i < p_{i+1}$ for $0 \leq i \leq k$, and set $n' = n - a_k m_k$. By Lemmas 2 and 3,

$$\begin{aligned} S_n(x; f) - f(x) &= \int_G (f(x-t) - f(x)) D_n(t)\,dt \\ &= \int_G (f(x-t) - f(x))(1 + \chi_{m_k}(t) + \cdots + \chi_{m_k}^{a_k - 1}(t)) D_{m_k}(t)\,dt \\ &\quad + \int_G (f(x-t) - f(x)) \chi_{m_k}^{a_k}(t) D_{n'}(t)\,dt = A + B. \end{aligned}$$

By Lemma 1,

$$A \leq m_k \int_{G_k} |f(x-t) - f(x)|\, |1 + \chi_{m_k}(t) + \cdots + \chi_{m_k}^{a_k - 1}(t)|\, dt$$

$$\leq m_k \frac{1}{m_k} \theta_k(f)\, a_k < p\, \theta_k(f),$$

and the last member tends to 0 (as $k \to \infty$) uniformly with respect to $x \in G$.

In order to find an estimate for B, we observe that if $t \in G_{k+1}$, $0 \leq \alpha < m_k$, and $0 \leq j < p_{k+1}$, then

$$D_{n'}(z_\alpha^{(k)} + jx_k + t) = \sum_{i=0}^{n'-1} \chi_i(z_\alpha^{(k)})\, \chi_i(jx_k + t) = \sum_{i=0}^{n'-1} \chi_i(z_\alpha^{(k)}) = D_{n'}(z_\alpha^{(k)}).$$

Also, for $t \in G_{k+1}$,

$$\sum_{j=0}^{p_{k+1}-1} \chi_{m_k}^{a_k}(jx_k + t) = \sum_{j=0}^{p_{k+1}-1} \chi_{m_k}^{a_k}(jx_k) = \sum_{j=0}^{p_{k+1}-1} (\omega_k^j)^{a_k} = 0,$$

where $\omega_k = e^{2\pi i / p_{k+1}}$. Consequently,

$$\int_G f(x)\, \chi_{m_k}^{a_k}(t)\, D_{n'}(t)\, dt = \sum_{\alpha=0}^{m_k-1} \sum_{j=0}^{p_{k+1}-1} \int_{z_\alpha^{(k)} + jx_k + G_{k+1}} f(x)\, \chi_{m_k}^{a_k}(t)\, D_{n'}(t)\, dt$$

$$= \frac{1}{m_{k+1}} f(x) \sum_{\alpha=0}^{m_k-1} D_{n'}(z_\alpha^{(k)})\, \chi_{m_k}^{a_k}(z_\alpha^{(k)}) \sum_{j=0}^{p_{k+1}-1} \chi_{m_k}(jx_k) = 0.$$

Thus

$$B = \int_G f(x-t)\, \chi_{m_k}^{a_k}(t)\, D_{n'}(t)\, dt = \int_{G_{k+1}} D_{n'}(0)\, \chi_{m_k}^{a_k}(0) \sum_{j=0}^{p_{k+1}-1} f(x - jx_k - t)\, \omega_k^{ja_k}\, dt$$

$$+ \int_{G_{k+1}} \sum_{\alpha=1}^{m_k-1} D_{n'}(z_\alpha^{(k)})\, \chi_{m_k}^{a_k}(z_\alpha^{(k)}) \sum_{j=0}^{p_{k+1}-1} f(x - z_\alpha^{(k)} - jx_k - t)\, \omega_k^{ja_k}\, dt = B_1 + B_2.$$

In what follows, we assume that f is real-valued. (In case f is complex-valued, the proof needs some obvious modifications.) For a real number a, we set $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$. Then, since

$$\sum_{j=0}^{p_{k+1}-1} \omega_k^{ja_k} = 0,$$

there exists an s, with $0 < s < p_{k+1}$, such that

$$\sum_{j=0}^{p_{k+1}-1} (\Re\, \omega_k^{ja_k})^+ = - \sum_{j=0}^{p_{k+1}-1} (\Re\, \omega_k^{ja_k})^- = s.$$

Therefore, denoting $f(x - jx_k - t)$ by f_j for $t \in G_{k+1}$, we have the relations

$$\left|\Re \sum_{j=0}^{p_{k+1}-1} f_j \omega_k^{ja_k}\right| = s\left|\sum_{j=0}^{p_{k+1}-1} f_j \frac{1}{s}(\Re\,\omega_k^{ja_k})^+ + \sum_{j=0}^{p_{k+1}-1} f_j \frac{1}{s}(\Re\,\omega_k^{ja_k})^-\right| \tag{1}$$

$$\leq s(\sup_j(f_j) - \inf_j(f_j)) \leq p_{k+1}\,\mathrm{osc}\,(f, x + G_k) \leq p_{k+1}\,\theta_k(f).$$

A similar argument shows that

$$\left|\Im \sum_{j=0}^{p_{k+1}-1} f_j \omega_k^{ja_k}\right| \leq p_{k+1}\,\theta_k(f). \tag{2}$$

Hence,

$$|B_1| \leq n'\frac{1}{m_{k+1}} 2p_{k+1}\,\theta_k(f) \leq 2\theta_k(f) = o(1) \qquad \text{as } k \to \infty.$$

Using the result of Lemma 5 and the assumption of the theorem, we find that

$$|B_2| \leq (p+1)m_k \int_{G_{k+1}} \sum_{\alpha=1}^{m_k-1} \frac{1}{\alpha}\left|\sum_{j=0}^{p_{k+1}-1} f(x - z_\alpha^{(k)} - jx_k - t)\omega_k^{ja_k}\right| dt$$

$$= (p+1)m_k \frac{1}{m_{k+1}} o(1) = o(1) \qquad \text{as } k \to \infty.$$

4. COROLLARIES OF THE MAIN THEOREM

In this section, we prove several consequences of Theorem 1, similar to results Salem obtained in the case of trigonometric Fourier series. Throughout the section, we assume that G satisfies condition (P).

COROLLARY 1. *Let f be a continuous function on G. If*

$$\lim \sum_{\alpha=1}^{m_k-1} \frac{1}{\alpha}\,\mathrm{osc}\,(f, x - z_\alpha^{(k)} + G_k) = 0$$

uniformly in $x \in G$, then the Fourier series of f converges uniformly on G.

Proof. We assume that f is real-valued. An argument similar to that used to prove inequalities (1) and (2) shows that

$$\left|\sum_{j=0}^{p_{k+1}-1} f(x - z_\alpha^{(k)} - jx_k)\,\omega_k^{ja_k}\right| \leq 2p\,\mathrm{osc}\,(f, x - z_\alpha^{(k)} + G_k)$$

for each k, each $a_k \in \{1, 2, \cdots, p_{k+1} - 1\}$, each α such that $1 \leq \alpha < m_k$, and each $x \in G$. Therefore,

$$\sum_{\alpha=1}^{m_k-1} \frac{1}{\alpha} \left| \sum_{j=0}^{p_{k+1}-1} f(x - z_\alpha^{(k)} - jx_k)\, \omega_k^{ja_k} \right|$$

$$\leq 2p \sum_{\alpha=1}^{m_k-1} \frac{1}{\alpha} \operatorname{osc}(f, x - z_\alpha^{(k)} - jx_k) = o(1) \quad \text{as } k \to \infty,$$

uniformly in $x \in G$, and $a_k \in \{1, 2, \cdots, p_{k+1} - 1\}$; that is, we can apply Theorem 1.

The next result was proved directly in [7, Section 3.5]; it is the analogue of the well-known Dini-Lipschitz test for trigonometric Fourier series.

COROLLARY 2. *Let* f *be a continuous function on* G *for which* $\theta_k(f) = o(k^{-1})$ *as* $k \to \infty$. *Then the Fourier series of* f *converges uniformly on* G.

Proof. We first observe that $m_k \leq p^k$ implies that $\log m_k = O(k)$. Furthermore,

$$\sum_{\alpha=1}^{m_k-1} \frac{1}{\alpha} \operatorname{osc}(f, x - z_\alpha^{(k)} + G_k) \leq \theta_k(f) \sum_{\alpha=1}^{m_k-1} \frac{1}{\alpha} = \theta_k(f)\, O(\log m_k) = o(1) \quad \text{as } k \to \infty .$$

Consequently, we can apply Corollary 1.

Using the ordering of G as defined in Section 1, Vilenkin defined the concept of bounded variation (BV) in the usual way [7, Section 3.2]. We generalize this concept in two ways.

Definition 4. A function f on G is of bounded fluctuation ($f \in BF$) if there exists a constant $F < \infty$ such that $\sum_{i=1}^{n} \operatorname{osc}(f, F_i) < F$ for each finite, disjoint collection $\{F_1, \cdots, F_n\}$ in which each F_i is a coset of some $G_{m(i)}$ and $\bigcup_{i=1}^{n} F_i = G$.

Definition 5. A function f on G is of generalized bounded fluctuation ($f \in GBF$) if there exists a constant $F_0 < \infty$ such that

$$\sum_{\alpha=0}^{m_n-1} \operatorname{osc}(f, z_\alpha^{(n)} + G_n) < F_0 \quad \text{for each } n .$$

We shall denote the smallest such constant by $F_0(f)$.

It is obvious that for continuous functions f on G, $f \in BV$ implies $f \in BF$ and $f \in BF$ implies $f \in GBF$. The converse of the last statement is not true, as can be seen from the following example of a continuous function on $G = \prod_{n=1}^{\infty} (Z(2))_n$:

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in z_{2^n-2}^{(n)} + G_n \ \ (n \text{ odd}), \\ 0 & \text{if } x \in z_{2^n-2}^{(n)} + G_n \ \ (n \text{ even}), \\ 0 & \text{if } x = (1, 1, \cdots). \end{cases}$$

In [8], J. L. Walsh proved that the Fourier series of a continuous function of bounded variation on $G = \prod_{n=1}^{\infty} (Z(2))_n$ converges uniformly. Corollary 3 is a generalization of this result.

COROLLARY 3. *If* f *is a continuous function on* G, *and if* $f \in GBF$, *then the Fourier series of* f *converges uniformly on* G.

Proof. Let $\{r(n)\}$ be an increasing sequence of natural numbers for which (i) $r(n) \to \infty$ as $n \to \infty$, (ii) $r(n) < m_n - 1$ for all n, and (iii) $\theta_n(f) \log r(n) \to 0$ as $n \to \infty$. Then

$$\sum_{\alpha=1}^{m_k-1} \frac{1}{\alpha} \operatorname{osc}(f, x - z_\alpha^{(k)} + G_k) \leq \theta_k(f) \sum_{\alpha=1}^{r(k)} \frac{1}{\alpha} + \sum_{\alpha=r(k)+1}^{m_k-1} \frac{1}{\alpha} \operatorname{ocs}(f, x - z_\alpha^{(k)} + G_k)$$

$$\leq \theta_k(f)\, O(\log r(k)) + \frac{1}{r(k)+1} F_0(f) = o(1) \quad \text{as } k \to \infty.$$

Therefore, Corollary 1 can be applied.

Finally, we prove an extension of Corollary 3. Let $\phi(u)$ be a continuous, real-valued, strictly increasing function, defined for $u \geq 0$, such that $\phi(0) = 0$ and $\lim_{u \to \infty} \phi(u) = \infty$. Let $\psi(u)$ be the inverse of $\phi(u)$. Next, let

$$\Phi(u) = \int_0^u \phi(t)\, dt \quad \text{and} \quad \Psi(u) = \int_0^u \psi(t)\, dt .$$

Functions Φ and Ψ thus obtained are called *complementary in the sense of W. H. Young*, and they satisfy the following inequality, due to W. H. Young, see [9, p. 16]:

$$\text{if } a, b \geq 0, \text{ then } ab \leq \Phi(a) + \Psi(b).$$

Definition 6. A function f on G is of generalized bounded Φ-fluctuation if there exists an $M < \infty$ such that

$$\sum_{i=0}^{m_n-1} \Phi(\operatorname{osc}(f, z_i^{(n)} + G_n)) < M \qquad \text{for all } n .$$

COROLLARY 4. *Let* Φ *and* Ψ *be functions complementary in the sense of W. H. Young, and let* $\sum_{k=1}^{\infty} \Psi(k^{-1}) < \infty$. *Then the Fourier series of every continuous function* f *on* G, *that has bounded* Φ*-fluctuation converges uniformly on* G.

Proof. Since $\sum_{k=1}^{\infty} \Psi(k^{-1}) < \infty$, we can find a decreasing sequence $\{\varepsilon(k)\}$ of positive numbers such that $\lim_{k \to \infty} \varepsilon(k) = 0$ and $\sum_{k=1}^{\infty} \Psi((k\varepsilon(k))^{-1}) < \infty$. Using Young's inequality, we see that there exists a constant $N < \infty$ such that

$$\sum_{\alpha=m}^{m_k-1} \operatorname{osc}(f, x - z_\alpha^{(k)} + G_k) \cdot \frac{1}{\alpha\,\varepsilon(\alpha)}$$

$$\leq \sum_{\alpha=m}^{m_k-1} \Phi(\operatorname{osc}(f, x - z_\alpha^{(k)} + G_k)) + \sum_{\alpha=m}^{m_k-1} \Psi((\alpha\,\varepsilon(\alpha))^{-1}) < N$$

for each $x \in G$ and each $m < m_n$.

Hence, if we choose a sequence $\{r(n)\}$ as in the proof of the previous corollary, we find that

$$\sum_{\alpha=1}^{m_k-1} \frac{1}{\alpha} \operatorname{osc}(f, x - z_\alpha^{(k)} + G_k) \leq \theta_k(f) \sum_{\alpha=1}^{r(k)} \frac{1}{\alpha} + \sum_{\alpha=r(k)+1}^{m_k-1} \frac{1}{\alpha} \operatorname{osc}(f, x - z_\alpha^{(k)} + G_k)$$

$$\leq \theta_k(f)\, O(\log r(k)) + N\varepsilon(r(k)+1) = o(1) \quad \text{as } k \to \infty.$$

Thus, Corollary 1 can again be applied.

REFERENCES

1. N. K. Bary, *A treatise on trigonometric series.* Vols. I, II. MacMillan, New York, 1964.
2. H. E. Chrestenson, *A class of generalized Walsh functions.* Pacific J. Math. 5 (1955), 17-31.
3. N. J. Fine, *On the Walsh functions.* Trans. Amer. Math. Soc. 65 (1949), 372-414.
4. E. Hewitt and K. A. Ross, *Abstract harmonic analysis.* Vol. I. Springer-Verlag, Berlin, 1963.
5. C. W. Onneweer, *On uniform convergence for Walsh-Fourier series.* Pacific J. Math. 34 (1970), 117-122.
6. R. Salem, *Essais sur les séries trigonométriques.* Actualités Sci. Indust., No. 862. Hermann, Paris, 1940.
7. N. Ja. Vilenkin, *On a class of complete orthonormal systems.* Amer. Math. Soc. Transl. (2) 28 (1963), 1-35.
8. J. L. Walsh, *A closed set of normal orthogonal functions.* Amer. J. Math. 45 (1923), 5-24
9. A. Zygmund, *Trigonometric series.* 2nd ed. Vol. I. Cambridge University Press, New York, 1959.

University of New Mexico
Albuquerque, New Mexico 87106
and
Syracuse University
Syracuse, New York 13210