AMBIGUOUS POINTS OF HOLOMORPHIC FUNCTIONS OF SLOW GROWTH

R. L. Hall

1. INTRODUCTION

Let f denote a holomorphic function in the open unit disc D. An *arc at* $e^{i\theta}$ is a curve $J \subset D$ such that $J \cup \{e^{i\theta}\}$ is a Jordan arc. The complex number a ($a = \infty$ is admitted) is an *asymptotic value* of f at $e^{i\theta}$ if there exists an arc at $e^{i\theta}$ on which f has the limit a at $e^{i\theta}$. Let $\Gamma(f, e^{i\theta})$ denote the set of asymptotic values of f at $e^{i\theta}$. If $\Gamma(f, e^{i\theta})$ contains at least two values, then $e^{i\theta}$ is called an *ambiguous point* of f.

It follows from the work of E. Lindelöf [8] that an f omitting two finite values has no ambiguous points (for a generalization of this result, see [7]). However, a result of W. Gross [5] can be used to show that even if f omits only one finite value, $\Gamma(f, 1)$ may nevertheless contain every complex number. By F. Bagemihl's ambiguous-point theorem [1], the set of ambiguous points of any f is at most countable. Ambiguous points of various classes of functions have been studied in [2], [3], [4], and [10].

Suppose $a \in \Gamma(f, e^{i\theta})$, and let J be an arc at $e^{i\theta}$ on which f has the limit a at $e^{i\theta}$. For each $\varepsilon > 0$, let $G(a, J, \varepsilon)$ denote the component of $\{z: |f(z) - a| < \varepsilon\}$ (of $\{z: |f(z)| > \varepsilon^{-1}\}$ if $a = \infty$) such that $G(a, J, \varepsilon) \cap J$ contains an arc at $e^{i\theta}$. The collection $\{G(a, J, \varepsilon): \varepsilon > 0\}$ is called the *tract* (or asymptotic tract) *of* f *at* $e^{i\theta}$ associated with the asymptotic value a and determined by J. Let

$$T_a = \{G(a, J, \varepsilon): \varepsilon > 0\} \quad \text{and} \quad T_b = \{G(b, J', \varepsilon): \varepsilon > 0\}$$

be tracts of f at $e^{i\theta}$. Then T_a and T_b are *distinct* if there exists an $\varepsilon > 0$ such that $G(a, J, \varepsilon) \cap G(a, J', \varepsilon) = \emptyset$. Note that the tracts are automatically distinct if $a \neq b$. However, more than one tract may be associated with an element $a \in \Gamma(f, e^{i\theta})$.

Let $n_*(f, e^{i\theta})$ ($n_\infty(f, e^{i\theta})$) denote the cardinal number of the set of tracts of f at $e^{i\theta}$ associated with finite (infinite) asymptotic values. For $0 < r < 1$, let $M(f, r)$ denote the maximum modulus of f on the circle $\{z: |z| = r\}$, and for $x > 0$, let $\log^+ x = \max(\log x, 0)$. G. R. MacLane [10, p. 54] has obtained the following results:

(A) *if* $\int_0^1 \log^+ M(f, r)\,dr < \infty$, *then*

$$n_*(f, e^{i\theta}) \leq 1 \quad \textit{and} \quad n_\infty(f, e^{i\theta}) \leq 2 \qquad \text{for each } \theta\,;$$

Received February 12, 1970.

This work was supported in part by a grant from the Wisconsin Alumni Research Foundation.

Michigan Math. J. 18 (1971).

(B) *if* $\int_0^1 (1 - r)\log^+ M(f, r)\,dr < \infty$, *then*

$$n_*(f, e^{i\theta}) \leq 2 \quad \textit{and} \quad n_\infty(f, e^{i\theta}) \leq 3 \qquad \text{for each } \theta.$$

2. STATEMENT OF RESULTS

THEOREM 1. *Suppose* f *is holomorphic in* D, *omits one finite value, and satisfies the condition*

$$(1) \qquad (1 - r)\log^+ M(f, r) = o(1) \qquad \text{as } r \to 1.$$

Then, for each θ,

$$n_*(f, e^{i\theta}) \leq 1 \quad \textit{and} \quad n_\infty(f, e^{i\theta}) \leq 2.$$

Remark. Example 1 in Section 4 shows that the same conclusion can not be obtained if the o(1) in (1) is replaced by O(1).

THEOREM 2. *Suppose* f *is holomorphic in* D, *omits* 0, *and satisfies the condition*

$$(2) \qquad (1 - r)\log M\left(f + \frac{1}{f}, r\right) = o(1) \qquad \textit{as } r \to 1.$$

Then, for each θ,

$$(3) \qquad n_*(f, e^{i\theta}) \leq 1 \quad \textit{and} \quad n_\infty(f, e^{i\theta}) \leq 1.$$

Moreover, if $a \in \Gamma(f, e^{i\theta})$ *and* $0 < |a| < \infty$, *then* f *has only one tract at* $e^{i\theta}$.

Remarks. Example 1 in Section 4 shows that the conclusion (3) can not be obtained if the o(1) in (2) is replaced by O(1). Example 2 in Section 4 shows that the conclusion (3) is in some sense best possible under a growth condition on $M(f + 1/f, r)$.

THEOREM 3. *Let* $\mu(r)$ *be a positive, increasing, continuous function for* $0 \leq r < 1$ *such that* $\mu(r) \to \infty$ *as* $r \to 1$. *Then there exists a function* f, *continuous in* $\overline{D} - \{1\}$ (*the bar denotes closure*) *and holomorphic in* D, *such that*

$$f \textit{ omits } 0,$$

$$M(f, r) < \mu(r) \quad \textit{for } 0 < r < 1,$$

$$n_*(f, 1) = 1 \quad \textit{and} \quad n_\infty(f, 1) = 2.$$

Remark. Compare Theorem 3 and Theorem 1.

THEOREM 4. *Let* $\mu(r)$ *satisfy the conditions in Theorem* 3, *and let* E *be a countable subset of the unit circle* C. *Then there exists a function* f, *holomorphic in* D, *such that*

$$M(f, r) < \mu(r) \qquad (0 < r < 1),$$

and such that

$$n_*(f, e^{i\theta}) = 1 \quad \textit{and} \quad n_\infty(f, e^{i\theta}) = 2$$

for each $e^{i\theta} \in E$.

Remarks. A consequence of Theorem 4 is that there exist *holomorphic* functions of arbitrarily slow growth with prescribed ambiguous points. Also, the conclusion of MacLane's result (A) is best possible.

3. PROOFS

Proof of Theorem 1. Assume, without loss of generality, that f omits 0. It suffices to show that $n_*(f, 1) \leq 1$ and $n_\infty(f, 1) \leq 2$. In the following two paragraphs, we shall show that if $n_*(f, 1) \geq 2$ or $n_\infty(f, 1) \geq 3$, then there exist a constant $K > 0$ and a component Δ of $\{z: |f(z)| > K\}$ such that $\overline{\Delta} \cap C = \{1\}$. From this we shall then obtain a contradiction of (1).

If $n_*(f, 1) \geq 2$, then there exist Jordan arcs J_1 and J_2 in $D \cup \{1\}$, with endpoints 0 and 1, such that the restriction of f to J_i has a finite limit at 1 (i = 1, 2) and the tracts determined by J_1 and J_2 are distinct. It can be assumed that J_1 and J_2 intersect only at 0 and 1. Since the tracts are distinct, it follows from the theorem of Gross and Iversen [11, p. 24] that f is unbounded in the bounded Jordan domain determined by $J_1 \cup J_2$. Since f is bounded on $J_1 \cup J_2$, there exists a number $K > 0$ such that some component Δ of $\{z: |f(z)| > K\}$ satisfies the condition $\overline{\Delta} \cap C = \{1\}$.

If $n_\infty(f, 1) \geq 3$, then there exist Jordan arcs J_1, J_2, and J_3 in $D \cup \{1\}$, with endpoints 0 and 1, such that the restriction of f to J_i has the limit ∞ at 1 (i = 1, 2, 3) and the corresponding tracts are distinct. It can be assumed that $J_i \cap J_k = \{0, 1\}$ if $1 \leq i < k \leq 3$. One of these arcs, say J_1, must lie (except for its endpoints) in the bounded Jordan domain B determined by the other two. Since the arcs determine distinct tracts, there exists an $\varepsilon > 0$ such that the components $G(\infty, J_i, \varepsilon)$ (i = 1, 2, 3) are pairwise disjoint. Let H denote the component of

$$B \cap (G(\infty, J_2, \varepsilon))^c \cap (G(\infty, J_3, \varepsilon))^c$$

that contains J_1 (S^c denotes the complement of S). The boundary of H consists of subsets of J_2 and J_3 that are compact in D, of a subset of $\{z: |f(z)| = \varepsilon^{-1}\}$, and of $\{1\}$. It follows that f is bounded on the portion of the boundary of H contained in D, and that f is unbounded in H. Thus there exist a number $K > 0$ and a component Δ of $\{z: |f(z)| > K\}$ such that $\overline{\Delta} \cap C = \{1\}$.

If F is a fractional linear transformation of D onto itself, then the composition f(F) satisfies the hypotheses of Theorem 1. Also,

$$n_*(f(F), F(e^{i\theta})) = n_*(f, e^{i\theta}) \quad \text{and} \quad n_\infty(f(F), F(e^{i\theta})) = n_\infty(f, e^{i\theta})$$

for each θ. Thus, we may assume that $0 \in \Delta$. Note that Δ is simply connected, since f omits 0, and let z(w) be a one-to-one conformal mapping of $\{w: |w| < 1\}$ onto Δ such that $z(0) = 0$.

Since f satisfies (1), it follows from the work of MacLane [10, p. 36] that for each $\lambda > 0$ the supremum of the diameters of the components of

$$\{z: |f(z)| = \lambda\} \cap \{z: r < |z| < 1\}$$

tends to zero as r tends to 1. Thus the boundary of Δ interior to D consists of arcs of the level set $\{z: |f(z)| = K\}$ that become Jordan curves when we adjoin the point 1. It follows that z(w) has a continuous extension to $\{w: |w| = 1\}$.

The set $\{w: |w| = 1 \text{ and } z(w) = 1\}$ has measure 0, by a version of Löwner's lemma [11, p. 34]. Thus the positive harmonic function $u(w) = \log\{K^{-1}|f(z(w))|\}$ has the radial limit zero almost everywhere on $\{w: |w| = 1\}$. If u has the radial limit ∞ at a point $e^{i\theta}$, then $z(e^{i\theta}) = 1$ and z(w) maps the radius to $e^{i\theta}$ into an arc at 1 on which f has the asymptotic value ∞. Furthermore, if u has the radial limit ∞ at $e^{i\theta_1}$ and $e^{i\theta_2}$ ($e^{i\theta_1} \neq e^{i\theta_2}$), then the images of the radii to $e^{i\theta_1}$ and $e^{i\theta_2}$ determine distinct tracts of f at 1. Since f satisfies (1), it follows from MacLane's result (B) (see the Introduction) that $n_\infty(f, 1) \leq 3$. Thus u has the radial limit ∞ in at most three distinct points.

Together with the properties of u obtained in the preceding paragraph, a theorem of Lohwater [9] implies that there exist nonnegative numbers c_1, c_2, and c_3 such that

$$u(\rho e^{i\theta}) = \sum_{j=1}^{3} c_j P(\rho, \theta - \theta_j), \tag{4}$$

where $w = \rho e^{i\theta}$ and

$$P(\rho, \theta - \alpha) = \frac{1 - \rho^2}{1 - 2\rho\cos(\theta - \alpha) + \rho^2}.$$

The function f is nonconstant under the assumption that either $n_*(f, 1) \geq 2$ or $n_\infty(f, 1) \geq 3$, so that at least one of the c_j, say c_1, is positive. By the lemma of Schwarz, $|z(w)| < |w|$. Thus it follows from (4) that

$$[1 - |z(\rho e^{i\theta_1})|]\log\{K^{-1}|f(z(\rho e^{i\theta_1}))|\} > (1 - \rho)u(\rho e^{i\theta_1}) > c_1. \tag{5}$$

But $z(\rho e^{i\theta_1}) \to 1$ as $\rho \to 1$, since $u(\rho e^{i\theta_1}) \to \infty$ as $\rho \to 1$. Therefore (5) contradicts (1). This completes the proof of Theorem 1.

Proof of Theorem 2. For each complex number $z \neq 0$, $|z| \leq 1 + |z + z^{-1}|$. Thus it follows from (2) that both f and 1/f satisfy the hypotheses of Theorem 1. The conclusion follows from Theorem 1 and the observation that the transformation $w \to w^{-1}$ effects a one-to-one correspondence between $\Gamma(f, e^{i\theta})$ and $\Gamma(1/f, e^{i\theta})$, for each θ.

Proof of Theorem 3. Let $w(z) = (1 + z)/(1 - z)$, and let $g(z) = w(z)e^{-w(z)}$. Then g is holomorphic except at $z = 1$. Also, the restriction of g to D has radial limit 0 at 1, and for each $u > 0$, g(z) tends to ∞ as z tends to 1 along the circle $\{z: \Re(w(z)) = u\}$. In fact, $(1 - r)M(r) = O(1)$, so that $n_*(g, 1) = 1$ and $n_\infty(g, 1) = 2$, by result (A) of MacLane (see the Introduction).

For each r $(0 < r < 1)$, let m(r) denote the minimum of M(g, r) and $\mu(r)$. Then m(r) is an increasing, continuous, and unbounded function for $0 < r < 1$. Since $g(\bar{z}) = \overline{g(z)}$ $(z \neq 1)$ and g(r) is bounded for real r $(|r| < 1)$, there exists a number r_0 $(0 < r_0 < 1)$ such that $g(r) < m(r)$ for $r_0 \leq |r| < 1$.

For each r $(r_0 \leq r < 1)$, let

$$\theta(r) = \inf\{\theta: 0 < \theta < \pi,\ |g(re^{i\theta})| = m(r)\}.$$

Since g is continuous, $|g(re^{i\theta(r)})| = m(r)$. Thus $|g(re^{i\theta(r)})| \to m(r')$ as $r \to r'$ $(r_0 < r' < 1)$, by the continuity of $m(r)$. Now the continuity of $\theta(r)$ follows from the observation that for each r $(r_0 \leq r < 1)$, there exists a number $\phi(r)$ $(0 < \phi(r) < \pi)$ such that $|g(re^{i\theta})|$, as a function of θ, is increasing for $0 \leq \theta \leq \phi(r)$ and decreasing for $\phi(r) \leq \theta \leq \pi$. Since $g(re^{i\theta(r)}) \to \infty$ as $r \to 1$ and g is holomorphic except at 1, it also follows that $re^{i\theta(r)} \to 1$ as $r \to 1$.

Let $G = \{z: |z| < r_0\} \cup \{z: |z| \geq r_0, |\arg z| < \theta(|z|)\}$. Let $z(t)$ be a one-to-one conformal mapping of $\{t: |t| < 1\}$ onto G such that $z(0) = 0$ and $z(1) = 1$. Note that $z(t)$ has a continuous extension to $\{t: |t| = 1\}$.

Choose a k $(0 < k < 1)$ such that $kM(g, r_0) \leq \min\{\mu(r): 0 \leq r \leq r_0\}$, and let $f(t) = kg(z(t))$. Then clearly f omits 0, f is continuous on $\overline{D} - \{1\}$, $n_*(f, 1) = 1$, and $n_\infty(f, 1) = 2$.

For $0 < r < 1$, let $G_r = G \cap \{z: |z| < r\}$. By the lemma of Schwarz, $|z(t)| < |t|$, so that $\{z(t): |t| < r\} \subset G_r$ for $0 < r < 1$. Then $M(f, r) < \mu(r)$ for $0 < r \leq r_0$, by the maximum principle and the choice of k. For $r_0 < r < 1$, the boundary of G_r consists of the following sets: an arc of $\{z: |z| = r_0\}$, the radial segment $\{\rho e^{i\theta(\rho)}: r_0 \leq \rho \leq r\}$, an arc of $\{z: |z| = r\}$, and the segment $\{\rho e^{-i\theta(\rho)}: r_0 \leq \rho \leq r\}$. Since $g(\bar{z}) = \overline{g(z)}$ and μ is increasing, it follows from the construction that $|g(z)| \leq \mu(r)$ for all z on the boundary of G_r. Since $0 < k < 1$, it follows from the maximum principle that $M(f, r) < \mu(r)$ $(r_0 < r < 1)$. This completes the proof of Theorem 3.

Proof of Theorem 4. Let h be a function of the type described in Theorem 3. Since $n_\infty(h, 1) = 2$, there exists a Jordan curve S contained in D (except for the point 1) such that the restriction of h to S has the limit ∞ at 1. Since h omits 0, the theorem of Gross and Iversen (applied to $1/h$) implies that the bounded Jordan domain determined by S contains an arc T at 1 such that the restriction of h to T has the limit 0 at 1. (This also follows from the construction given in the proof of Theorem 3.)

Let $E = \{e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_k}, \cdots\}$, where $e^{i\theta_j} \neq e^{i\theta_k}$ if $j \neq k$. For each $k \geq 1$, let

$$g_k(z) = h(e^{-i\theta_k} z), \qquad S_k = \{e^{i\theta_k} z: z \in S\}, \qquad T_k = \{e^{i\theta_k} z: z \in T\}.$$

Choose c_1 so that $0 < c_1 < 1/2$. If $k > 1$, it follows from the properties of h and the definition of g_k that g_k is continuous on the compact set

$$M_k = \bigcup_{j<k} (S_j \cup T_j). \tag{6}$$

For each $k > 1$, choose c_k so that

$$0 < c_k < 2^{-k} \tag{7}$$

and

$$c_k |g_k(z)| < 2^{-k} \quad \text{for all } z \in M_k. \tag{8}$$

By the definition of g_k, $M(g_k, r) = M(h, r) < \mu(r)$ for $0 < r < 1$. Thus, it follows from (7) that

$$M(c_k g_k, r) < 2^{-k}\mu(r) \quad (0 < r < 1,\ k = 1, 2, \cdots).$$

Therefore the series $\sum c_k g_k$ converges (uniformly on each compact subset of D) to a function f holomorphic in D. Furthermore,

$$M(f, r) < \mu(r) \quad (0 < r < 1).$$

For each $k \geq 1$, write $f = \phi_k + c_k g_k + \psi_k$, where

(9) $$\phi_k = 0 \quad \text{if } k = 1, \qquad \phi_k = \sum_{j=1}^{k-1} c_j g_j \quad \text{if } k > 1,$$

and

(10) $$\psi_k = \sum_{j=k+1}^{\infty} c_j g_j.$$

If $j \neq k$, the function g_j is finite-valued and continuous on $S_k \cup T_k$. Therefore, it follows from (9) that for each k the function ϕ_k is bounded and continuous on $S_k \cup T_k$. Also, it follows from (6), (8), and (10) that ψ_k is bounded and continuous on $S_k \cup T_k$, for each k. Since the restriction of $c_k g_k$ to T_k has the limit 0 at $e^{i\theta_k}$, the restriction of f to T_k has a finite limit at $e^{i\theta_k}$. Therefore,

(11) $$n_*(f, e^{i\theta_k}) \geq 1.$$

The restriction of f to $S_k - \{e^{i\theta_k}\}$ has the limit ∞ at $e^{i\theta_k}$, since the restriction of $c_k g_k$ to $S_k - \{e^{i\theta_k}\}$ has the limit ∞ at $e^{i\theta_k}$. Therefore,

(12) $$n_\infty(f, e^{i\theta_k}) \geq 2,$$

because f is bounded on T_k and the arc T_k joins 0 to $e^{i\theta_k}$ through the bounded Jordan domain determined by S_k. Now, if we use $\min(\mu(r), (1-r)^{-1})$ in place of $\mu(r)$ to obtain h from Theorem 3, then (A), (11), and (12) guarantee that $n_*(f, e^{i\theta_k}) = 1$ and $n_\infty(f, e^{i\theta_k}) = 2$. The proof of Theorem 4 is complete.

4. EXAMPLES

Example 1. Let $f = 1/g$, where g is the function introduced in the proof of Theorem 3. Then f is holomorphic in D, omits 0, and satisfies the condition

$$(1 - r)\log M(f, r) = O(1) \quad \text{as } r \to 1.$$

Also, f satisfies the condition

$$(1 - r)\log M\left(f + \frac{1}{f}, r\right) = O(1) \quad \text{as } r \to 1.$$

However, $n_*(f, 1) = 2$, since $n_\infty(g, 1) = 2$.

Example 2. Let $\mu(r)$ be a continuous, increasing, unbounded function for $0 < r < 1$ such that $\mu(r) > 2$ $(0 < r < 1)$. Let $U(z) = \Re(iw(z))$, where

$$w(z) = \frac{1+z}{1-z}.$$

Then U is harmonic in D and $U(\bar{z}) = -U(z)$ $(z \neq 1)$. Also, for each $k > 0$, U(z) has the limit $-\infty$ as z tends to 1 along the semicircle $\{z: \Im w(z) = k,\ \Im z > 0\}$, and U(z) has the limit ∞ as z tends to 1 along the semicircle $\{z: \Im w(z) = k,\ \Im z < 0\}$. We can apply the technique used in the proof of Theorem 3 to U, to obtain a function u, harmonic in D, with the properties

(13) u has the limit $-\infty$ along an arc at 1,

(14) u has the limit ∞ along an arc at 1,

and

(15) $$|u(z)| < \log \frac{\mu(|z|)}{2} \quad \text{for all } z \in D.$$

Let v be a harmonic function in D that is conjugate to u, and let $f = e^{u+iv}$. Then f is holomorphic in D and omits 0. Furthermore

$$\left| f + \frac{1}{f} \right| \leq e^{u} + e^{-u},$$

and thus, by (15),

$$M\left(f + \frac{1}{f}, r\right) < \mu(r) \qquad (0 < r < 1).$$

It follows from (13) and (14) that $0 \in \Gamma(f, 1)$ and $\infty \in \Gamma(f, 1)$.

REFERENCES

1. F. Bagemihl, *Curvilinear cluster sets of arbitrary functions.* Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 379-382.

2. F. Bagemihl and W. Seidel, *Functions of bounded characteristic with prescribed ambiguous points.* Michigan Math. J. 3 (1955-56), 77-81.

3. G. T. Cargo, *Almost-bounded holomorphic functions with prescribed ambiguous points.* Canad. J. Math. 16 (1964), 231-240.

4. F. W. Gehring, *The asymptotic values for analytic functions with bounded characteristic.* Quart. J. Math. Oxford Ser. (2) 9 (1958), 282-289.

5. W. Gross, *Eine ganze Funktion, für die jede komplexe Zahl Konvergenzwert ist.* Math. Ann. 79 (1918), 201-208.

6. R. L. Hall, *On the asymptotic behavior of functions holomorphic in the unit disc.* Math. Z. 107 (1968), 357-362.

7. O. Lehto and K. I. Virtanen, *Boundary behavior and normal meromorphic functions.* Acta Math. 97 (1957), 46-65.

8. E. Lindelöf, *Sur un principe général de l'analyse et ses applications à la théorie de la représentation conforme*. Acta Soc. Fennicae 46 (1915), Nr. 4, 35 pp.

9. A. J. Lohwater, *A uniqueness theorem for a class of harmonic functions*. Proc. Amer. Math. Soc. 3 (1952), 278-279.

10. G. R. MacLane, *Asymptotic values of holomorphic functions*. Rice Univ. Studies 49 (1963), no. 1, 83 pp.

11. K. Noshiro, *Cluster sets*. Ergebnisse der Mathematik und ihrer Grenzgebiete. N. F., Heft 28. Springer-Verlag, Berlin, 1960.

University of Wisconsin—Milwaukee
Milwaukee, Wisconsin 53201