INEQUALITIES GOVERNING THE OPERATOR RADII ASSOCIATED WITH UNITARY ρ-DILATIONS

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1. INTRODUCTION

Our purpose in this paper is to present some results concerning the operator radii $w_\rho(T)$. First we recall the pertinent definitions. Suppose T is an operator on a Hilbert space $\mathscr{H}$ (in what follows, all Hilbert spaces are complex, and all operators are bounded and linear). We say the operator T belongs to the class C_ρ $(0 < \rho < \infty)$ if there exists a unitary operator U on some Hilbert space $\mathscr{K}$ such that $\mathscr{K}$ contains $\mathscr{H}$ as a subspace and such that $T^n h = \rho P_{\mathscr{H}} U^n h$ for all $h \in \mathscr{H}$ $(n = 1, 2, \cdots)$. B. Sz.-Nagy and C. Foiaş introduced the classes C_ρ in [9] to provide a unified framework for two results that we may state as follows: (i) (see Sz.-Nagy [7]) $T \in C_1$ if and only if $\|T\| \leq 1$; (ii) (see C. A. Berger [1]) $T \in C_2$ if and only if $w(T) \leq 1$, where $w(T)$ denotes the numerical radius of T, that is,

$$w(T) = \sup\{|(Th, h)|: h \in \mathscr{H} \text{ and } \|h\| = 1\}.$$

In our paper [5], we defined the operator radii $w_\rho(\cdot)$ $(0 < \rho < \infty)$ by the equation

$$w_\rho(T) = \inf\{u: u > 0 \text{ and } u^{-1} T \in C_\rho\}.$$

Independently, J. P. Williams used the same functions in [11]. The family of operator radii $w_\rho(\cdot)$ includes the familiar operator norms $\|\cdot\|$ $(= w_1(\cdot))$ and $w(\cdot)$ $(= w_2(\cdot))$. We may adjoin the other well-known operator radius, namely, the spectral radius $\nu(\cdot)$, to this family in a natural way: the relation

$$\lim_{\rho \to \infty} w_\rho(T) = \nu(T)$$

holds, so that we are led to define $w_\infty(T)$ as $\nu(T)$.

These and other known properties of the classes C_ρ and the functions $w_\rho(\cdot)$ are described carefully in Section 2. Sections 3, 4, and 5 contain results that we believe to be new. Experience suggests that $w_\rho(T)$ may be a convex function of ρ (for fixed T) in every case. In Section 3, we obtain results in this direction. For example, we show that if $0 < \rho_1, \rho_2 < 2$ and $F(\rho) = (w(T))^{-1}$, then

$$F((\rho_1 + \rho_2)/2) \geq \lambda F(\rho_1) + (1 - \lambda) F(\rho_2),$$

where $\lambda = (2 - \rho_1)(2 - \rho_1 + 2 - \rho_2)^{-1}$. From this inequality, we deduce that the function $(2 - \rho)(w_\rho(T))^{-1}$ is increasing on $(0, 1]$. Combining this result with the same inequality, we show that $w_\rho(T)$ is indeed convex in the range $(0, 1]$. Section 4 contains convexity results of a less precise nature; there we simply demonstrate the existence of certain convexity constants. In Section 5, we apply the earlier

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results to the problem of finding nontrivial multiplicative inequalities of the form $w_{\rho\sigma}(TS) \leq K w_\rho(T) w_\sigma(S)$, under the assumption that T and S are commuting operators.

2. BASIC PROPERTIES OF $w_\rho(\cdot)$ AND C_ρ

We list the following results for later reference. Proofs are given in [5] (a number of the results have also appeared elsewhere; see, in particular, Williams [11], C. A. Berger and J. G. Stampfli [2], E. Durszt [4], and Sz.-Nagy [8]).

(2.0) For $\rho \in (0, \infty)$, the inequality $w_\rho(T) \leq 1$ holds if and only if $T \in C_\rho$.

(2.1) The function $w_\rho(T)$ is continuous for $\rho \in (0, \infty]$; in particular, $\nu(T) = w_\infty(T) = \lim_{\rho \to \infty} w_\rho(T)$.

(2.2) If $0 < \rho < \rho' < \infty$, then $w_{\rho'}(T) \leq w_\rho(T)$ and $w_\rho(T) \leq (2(\rho'/\rho) - 1) w_{\rho'}(T)$.

(2.3) For all $\rho \in (0, \infty]$, $w_\rho(T) \geq \rho^{-1} \|T\|$.

(2.4) $w_1(T) = \|T\|$.

(2.5) $w_2(T) = w(T)$.

(2.6) The function $w_\rho(\cdot)$ is homogeneous, that is, $w_\rho(zT) = |z| w_\rho(T)$ for each complex number z. Moreover, if $0 < \rho \leq 2$, then $w_\rho(\cdot)$ is a norm on the space $B(\mathscr{H})$ of operators on $\mathscr{H}$, that is, we also have the inequality

$$w_\rho(T + S) \leq w_\rho(T) + w_\rho(S) \qquad (T, S \in B(\mathscr{H})).$$

(2.7) For each $\rho \in (0, \infty]$, we have the inequality $w_\rho(T^n) \leq (w_\rho(T))^n$ $(n = 1, 2, \cdots)$. Of course, there is equality if $\rho = \infty$.

(2.8) If T and S are doubly commuting operators (that is, if $TS = ST$ and $T^*S = ST^*$), then

$$w_{\rho\sigma}(TS) \leq w_\rho(T) w_\sigma(S) \qquad (\rho, \sigma \in (0, \infty]).$$

(2.9) If T is a normal operator, then

$$w_\rho(T) = \begin{cases} \|T\| (2\rho^{-1} - 1) & \text{if } 0 < \rho < 1, \\ \|T\| & \text{if } 1 \leq \rho \leq \infty. \end{cases}$$

In fact, this is the case whenever T is normaloid, that is, whenever $\nu(T) = \|T\|$.

(2.10) For all $\rho \in (0, \infty]$, $w_\rho(T) \geq w_\rho(I) \nu(T)$.

(2.11) If $T^2 = 0$, then $w_\rho(T) = \rho^{-1} \|T\|$ for all $\rho \in (0, \infty]$.

(2.12) If $0 < \sigma < \rho \leq \infty$ and $w_\rho(T) = w_\sigma(T)$, then $w_{\rho'}(T) = w_\sigma(T)$ whenever $\sigma \leq \rho' \leq \infty$.

We shall also need the following criterion for membership in the class C_ρ $(0 < \rho < \infty)$.

(2.13) $T \in C_\rho \Longleftrightarrow \Re((I - zT)h, h) \geq (1 - (\rho/2)) \|(I - zT)h\|^2$ for each $h \in \mathscr{H}$ and each complex z such that $|z| \leq 1$.

This is the basic intrinsic characterization of elements of the class C_ρ; it is established in [9] under the additional assumption (for proving $(\Longleftarrow)$) that $\nu(T) \leq 1$. In [9], the authors point out that this assumption is redundant if $\rho \leq 2$; more recently, C. Davis [3, Prop. 8.3] showed that it is always redundant, so that (2.13) is true as it stands. We include a direct proof of this in the form of the following simple lemma.

LEMMA. *Suppose that* $\Re((I - zT)h, h) \geq K \|(I - zT)h\|^2$ *for some real number* K *and for all* $h \in \mathscr{H}$ *and all complex* z *such that* $|z| \leq 1$. *Then* $\nu(T) \leq 1$.

Proof. If $\nu(T)$ is greater than 1, then we can take some $\lambda \in \sigma(T)$ such that $|\lambda| = \nu(T) > 1$. Since λ is in the boundary of the spectrum $\sigma(T)$, λ must be an approximate eigenvalue for T, that is, there are $h_n \in \mathscr{H}$ such that $\|h_n\| = 1$ and $g_n = (Th_n - \lambda h_n) \to_n 0$. Let $0 < \varepsilon \leq |\lambda| - 1$ and $z = (1 + \varepsilon)\lambda^{-1}$, so that $(I - zT)h_n = -\varepsilon h_n - zg_n$. Our inequality (with $h = h_n$) then reads:

$$-\varepsilon - \Re(zg_n, h_n) \geq K(\varepsilon^2 + 2\Re(\varepsilon h_n, zg_n) + |z|^2 \|g_n\|^2).$$

Hence, letting $n \to \infty$, we find that $-\varepsilon \geq K\varepsilon^2$, so that $K \leq -\varepsilon^{-1}$. But this becomes false if we choose ε small enough. ■

3. NEAR-CONVEXITY OF $w_\rho(T)$ FOR $0 < \rho < 2$

In those few cases where we can compute $w_\rho(T)$ explicitly (see (2.9) and (2.11)), the function turns out to be convex in ρ. Furthermore, the natural operations that may be performed on these functions preserve convexity; for example, if $T, S \in B(\mathscr{H})$ and $T \oplus S$ is the operator acting on $\mathscr{H} \oplus \mathscr{H}$ in the obvious way, then $w_\rho(T \oplus S) = w_\rho(T) \vee w_\rho(S)$ (see Section 4). The theorems in the present section support the conjecture that $w_\rho(T)$ is convex in every case. Throughout the section, T denotes an operator in $B(\mathscr{H})$. We rule out the trivial case where $T = 0$ (in this case, of course, $w_\rho(T) = 0$ for all ρ). We may assume, therefore, that $w_\rho(T) > 0$ for all $\rho \in (0, \infty)$ (see (2.3)).

THEOREM 3.1. *Suppose* $0 < \rho_1 < \rho_2 < 2$, *let* $\rho_3 = (\rho_1 + \rho_2)/2$, *and let* $g(\rho) = (2 - \rho)(w_\rho(T))^{-1}$. *Then* $g(\rho_3) \geq (g(\rho_1) + g(\rho_2))/2$.

Proof. Since the operator radii are homogeneous, $w_\rho((w_\rho(T))^{-1}T) = 1$, so that $(w_\rho(T))^{-1}T \in C_\rho$. From (2.13) we obtain the inequality

$$\Re((I - z(w_\rho(T))^{-1}T)h, h) \geq \frac{1}{2}(2 - \rho)\|(I - z(w_\rho(T))^{-1}T)h\|^2, \tag{3.1}$$

whenever $h \in \mathscr{H}$, $|z| \leq 1$, and $\rho \in (0, \infty)$. When $2 - \rho > 0$, (3.1) is equivalent to the assertion

$$\Re\{((2 - \rho)I - zg(\rho)T)h, h\} \geq \frac{1}{2}\|((2 - \rho)I - zg(\rho)T)h\|^2. \tag{3.2}$$

Combining the inequalities obtained from (3.2) by putting $\rho = \rho_1$ and $\rho = \rho_2$, and recalling that

$$\|a\|^2 + \|b\|^2 \left(= \frac{1}{2}(\|a+b\|^2 + \|a-b\|^2)\right) \geq \frac{1}{2}\|a+b\|^2$$

for all a, b $\in \mathscr{H}$, we find that

$$\Re((2(2-\rho_3)I - z(g(\rho_1)+g(\rho_2))T)h, h) \geq \frac{1}{4}\|(2(2-\rho_3)I - z(g(\rho_1)+g(\rho_2))T)h\|^2 . \tag{3.3}$$

Since $2 - \rho_3 > 0$, it follows that

$$\begin{aligned}&\Re\left(\left(I - z(2-\rho_3)^{-1}\frac{1}{2}(g(\rho_1)+g(\rho_2))T\right)h, h\right)\\ &\qquad \geq \frac{1}{2}(2-\rho_3)\,\|(I - z(2-\rho_3)^{-1}\frac{1}{2}(g(\rho_1)+g(\rho_2))T)h\|^2 ,\end{aligned} \tag{3.4}$$

whenever $h \in \mathscr{H}$ and $|z| \leq 1$. From (2.13) we conclude that

$$(2-\rho_3)^{-1}\frac{1}{2}(g(\rho_1)+g(\rho_2))T \in C_{\rho_3} .$$

This implies that $w_{\rho_3}((2-\rho_3)^{-1}\frac{1}{2}(g(\rho_1)+g(\rho_2))T) \leq 1$, so that

$$\frac{1}{2}(g(\rho_1)+g(\rho_2)) \leq g(\rho_3) . \quad \blacksquare$$

We can rewrite the inequality of Theorem 3.1 so that it appears to be closer to the assertion that $w_\rho(T)$ is convex. Let $F(\rho) = (w_\rho(T))^{-1}$; we may interpret the following form of the inequality as saying that $F(\rho)$ is nearly concave in the range (0, 2).

COROLLARY 3.2. *If* $0 < \rho_1 < \rho_2 < 2$, *then* $F\left(\frac{1}{2}(\rho_1+\rho_2)\right) \geq \lambda_1 F(\rho_1) + \lambda_2 F(\rho_2)$, *where* $\lambda_i = (2-\rho_i)(2-\rho_1+2-\rho_2)^{-1}$ $(i = 1, 2)$.

We note that the result above yields a new proof of (2.12) in the range (0, 2]; this argument is quite different from that given in [5].

COROLLARY 3.3. *If* $0 < \rho_0 < 2$ *and if* $w_\rho(T) = w_{\rho_0}(T)$ *for some* $\rho > \rho_0$, *then* $w_\tau(T) = w_{\rho_0}(T)$ *whenever* $\tau \in [\rho_0, 2]$.

Proof. Let $\tau_0 = \max\{\tau: w_\tau(T) = w_{\rho_0}(T)\}$. Clearly, $\tau_0 > \rho_0$, and if we assume that $\tau_0 < 2$, then $\rho_1 = \tau_0 - \varepsilon \geq \rho_0$ and $\rho_2 = \tau_0 + \varepsilon < 2$ for small enough $\varepsilon > 0$. But then the inequality of Corollary 3.2 implies that $F(\tau_0) \geq \lambda_1 F(\tau_0 - \varepsilon) + \lambda_2 F(\tau_0 + \varepsilon)$, where $\lambda_1 + \lambda_2 = 1$ and $\lambda_2 > 0$. Since $F(\tau_0 - \varepsilon) = F(\tau_0) = F(\rho_0)$, we conclude that $\lambda_2 F(\rho_0) \geq \lambda_2 F(\tau_0 + \varepsilon)$, so that $F(\rho_0) = F(\tau_0 + \varepsilon)$ (of course, F is nondecreasing, by (2.2)). This is contradictory, by the definition of τ_0. $\blacksquare$

COROLLARY 3.4. *The function* $g(\rho) = (2-\rho)(w_\rho(T))^{-1}$ *is concave on* (0, 2].

Proof. By (2.1), $g(\rho)$ is continuous. It follows that we need only establish midpoint concavity. But this is the assertion of Theorem 3.1. $\blacksquare$

THEOREM 3.5. *The function* $g(\rho) = (2 - \rho)(w_\rho(T))^{-1}$ *is strictly increasing on* $(0, 1]$.

Proof. By Corollary 3.4, $g(\rho)$ is concave on $(0, 1]$. Thus it is sufficient to show that $g(\rho) < g(1)$ whenever $0 < \rho < 1$. By (2.3) and (2.4),

$$(w_\rho(T))^{-1} \leq \rho \|T\|^{-1} = \rho(w_1(T))^{-1}.$$

It follows that $g(\rho) \leq (2 - \rho)\rho(2 - 1)(w_1(T))^{-1} = (2\rho - \rho^2)g(1)$. But $1 - (2\rho - \rho^2) = (1 - \rho)^2 > 0$. ∎

THEOREM 3.6. *The function* $w_\rho(T)$ *is strictly convex on* $(0, 1]$.

Proof. Let $f(\rho)$ denote $w_\rho(T)$. By (2.1), $f(\rho)$ is continuous, so that it is sufficient to prove strict midpoint convexity. Thus we need only show that if $0 < \rho_1 < \rho_2 \leq 1$ and $\rho_3 = (\rho_1 + \rho_2)/2$, then $f(\rho_3) < \frac{1}{2}(f(\rho_1) + f(\rho_2))$. Setting $F(\rho) = 1/f(\rho)$, we may state the desired inequality in the form

$$F(\rho_3) > 2(f(\rho_1) + f(\rho_2))^{-1}. \tag{3.5}$$

By Corollary 3.2, we have the inequality

$$F(\rho_3) \geq \lambda F(\rho_1) + \mu F(\rho_2), \tag{3.6}$$

where $\lambda = (2 - \rho_1)(2 - \rho_1 + 2 - \rho_2)^{-1}$ and $\mu = (2 - \rho_2)(2 - \rho_1 + 2 - \rho_2)^{-1}$. We conclude that (3.5) will follow provided we establish that

$$\lambda F(\rho_1) + \mu F(\rho_2) > 2(f(\rho_1) + f(\rho_2))^{-1}, \tag{3.7}$$

or, equivalently (since $\lambda + \mu = 1$), that

$$\lambda(f(\rho_2)/f(\rho_1)) + \mu(f(\rho_1)/f(\rho_2)) > 1. \tag{3.8}$$

Now, for $x > 0$, the inequality $\lambda(1/x) + \mu x > 1$ is equivalent to the inequality $\mu x^2 - x + \lambda > 0$. But the roots of $\mu x^2 - x + \lambda = 0$ are 1 and λ/μ; note that $\lambda/\mu = (2 - \rho_1)/(2 - \rho_2) > 1$. It follows that (3.8) holds, provided that $f(\rho_1)/f(\rho_2) > (2 - \rho_1)/(2 - \rho_2)$, that is, provided that $g(\rho_2) > g(\rho_1)$. This is a consequence of Theorem 3.5. ∎

The final result of this section provides a near-convexity inequality for the range $(1, 2)$. We shall use a modified form of the criterion (2.13): if $1 < \rho < 2$, then

$$T \in C_\rho \iff \|zI - T\| \leq |z| + 1 \text{ for all complex } z \text{ such that } |z| \geq (\rho - 1)/(2 - \rho). \tag{3.9}$$

For the proof we refer to the book of Sz.-Nagy and Foiaş [10, p. 47].

THEOREM 3.7. *Suppose that* $1 < \rho_1 < \rho_2 < 2$, *and let* $\phi(\rho) = (\rho - 1)/(2 - \rho)$ *for* $\rho \in (1.2)$. *Then* $w_\rho(T) \leq \frac{1}{2}(w_{\rho_1}(T) + w_{\rho_2}(T))$ *whenever* $\rho \geq \phi^{-1}(\lambda_1 \phi(\rho_1) + \lambda_2 \phi(\rho_2))$, *where* $\lambda_i = w_{\rho_i}(T)(w_{\rho_1}(T) + w_{\rho_2}(T))^{-1}$ $(i = 1, 2)$.

Proof. Let $a_i = w_{\rho_i}(T)$ $(i = 1, 2)$. From (3.9) it follows that

(3.10) $\|z_i I - a_i^{-1}T\| \leq |z_i| + 1$ whenever $|z_i| \geq \phi(\rho_i)$ $(i = 1, 2)$.

Now suppose that $|z| \geq \lambda_1 \phi(\rho_1) + \lambda_2 \phi(\rho_2)$. Then $|z| = \lambda_1 b_1 + \lambda_2 b_2$ for some b_1 and b_2 such that $b_i \geq \phi(\rho_i)$ $(i = 1, 2)$. Let $z_i = b_i z/|z|$ $(i = 1, 2)$. From the inequalities of (3.10), it follows that

(3.11) $\|\lambda_1 z_1 I - \lambda_1 a_1^{-1} T\| + \|\lambda_2 z_2 I - \lambda_2 a_2^{-1} T\| \leq \lambda_1 |z_1| + \lambda_2 |z_2| + 1$.

Using the triangle inequality for the operator norm and (3.11), we see that

(3.12) $$\|zI - 2(a_1 + a_2)^{-1} T\| \leq |z| + 1 .$$

Let $\rho = \phi^{-1}(\lambda_1 \phi(\rho_1) + \lambda_2 \phi(\rho_2))$; we have shown that (3.12) holds whenever $|z| \geq \phi(\rho)$. By (3.9), $2(a_1 + a_2)^{-1} T \in C_\rho$, so that $w_\rho(T) \leq (a_1 + a_2)/2$. Since $w_\tau(T)$ is a nonincreasing function of τ, the theorem is proved. ■

COROLLARY 3.8. *If* $1 < \rho_1 < \rho_2 < 2$, *then* $w_\rho(T) \leq (w_{\rho_1}(T) + w_{\rho_2}(T))/2$, *provided* $(\rho_2 - \rho)(\rho_2 - \rho_1)^{-1} \leq \frac{1}{2}(2 - \rho_2)^2 (2 - \rho_1)^{-2}$.

Proof. We use the notation of Theorem 3.7. By (2.2), $\lambda_1 \geq \lambda_2$, so that $\lambda_1 \phi(\rho_1) + \lambda_2 \phi(\rho_2) \leq \frac{1}{2}(\phi(\rho_1) + \phi(\rho_2))$, since ϕ is increasing in $(1, 2)$. Let ρ_3 be such that $\phi(\rho_3) = \frac{1}{2}(\phi(\rho_1) + \phi(\rho_2))$. Since $\phi'(t) = (2 - t)^{-2}$ and is increasing,

$$\frac{1}{2}(\phi(\rho_2) - \phi(\rho_1))(\rho_2 - \rho_3)^{-1} \leq (2 - \rho_2)^{-2}$$

and

$$(\phi(\rho_2) - \phi(\rho_1))(\rho_2 - \rho_1)^{-1} \geq (2 - \rho_1)^{-2},$$

so that $(\rho_2 - \rho_3)(\rho_2 - \rho_1)^{-1} \geq \frac{1}{2}(2 - \rho_2)^2 (2 - \rho_1)^{-2}$. Thus, if our hypothesis concerning ρ is satisfied, then $\rho \geq \rho_3$. It follows that

$$\phi(\rho) \geq \phi(\rho_3) = \frac{1}{2}(\phi(\rho_1) + \phi(\rho_2)) \geq \lambda_1 \phi(\rho_1) + \lambda_2 \phi(\rho_2).$$

This implies that $\rho \geq \phi^{-1}(\lambda_1 \phi(\rho_1) + \lambda_2 \phi(\rho_2))$, so that Theorem 3.7 applies. ■

4. EXISTENCE OF CONVEXITY CONSTANTS FOR $w_\rho(\cdot)$

Outside the range (0, 2), the only general result hinting at convexity of the function $w_\rho(T)$ appears to be (2.12). In this section, however, we deduce stronger results of this type by combining (2.12) with some observations concerning direct sums of operators (Theorems 4.1, 4.2, and 4.3).

Whenever we have a sequence of operators $T_n \in B(\mathscr{H}_n)$ $(n = 1, 2, \cdots)$ such that $\sup_n \|T_n\| < \infty$, we may define an operator T on the direct sum $\mathscr{H} = \bigoplus_1^\infty \mathscr{H}_n$, as follows. An element h of $\mathscr{H}$ is determined by a sequence $\{h_n\}_1^\infty$, where

$h_n \in \mathscr{H}_n$ and $\sum_1^\infty \|h_n\|^2 < \infty$. We define Th to be $\{T_n h_n\}_1^\infty$; it is easy to see that T is linear and that $\|T\| = \sup_n \|T_n\|$. When T is constructed in this way, we shall use the notation $T = \bigoplus_1^\infty T_n$.

THEOREM 4.1. *If* $T_n \in B(\mathscr{H}_n)$ $(n = 1, 2, \cdots)$ *and* $\sup_n \|T_n\| < \infty$, *then* $w_\rho\left(\bigoplus_1^\infty T_n\right) = \sup_n w_\rho(T_n)$ *whenever* $0 < \rho < \infty$.

Proof. Let $a = \sup_n w_\rho(T_n)$. Then, for each n, $a^{-1} T_n \in C_\rho$, so that there exists a Hilbert space $\mathscr{K}_n$, containing $\mathscr{H}_n$ as a subspace, such that some unitary $U_n \in B(\mathscr{K}_n)$ satisfies the condition

$$(a^{-1} T_n)^k = \rho P_{\mathscr{H}_n} U_n^k \text{ on } \mathscr{H}_n \qquad (k = 1, 2, \cdots).$$

Now $U = \bigoplus_1^\infty U_n$ is unitary on $\mathscr{K} = \bigoplus_1^\infty \mathscr{K}_n$, and $\mathscr{H} = \bigoplus_1^\infty \mathscr{H}_n$ is imbedded in $\mathscr{K}$ in the obvious way. Let $T = \bigoplus_1^\infty T_n$. Clearly, $T^k = \bigoplus_1^\infty T_n^k$ and $U^k = \bigoplus_1^\infty U_n^k$, and it follows that $(a^{-1} T)^k = \rho P_{\mathscr{H}} U^k$ on $\mathscr{H}$ $(k = 1, 2, \cdots)$. Hence $a^{-1} T \in C_\rho$, so that $w_\rho(T) \leq a$.

On the other hand, it is clear (from (2.13), for example) that if $S \in B(\mathscr{H})$ and S has an invariant subspace $\mathscr{H}_0$, then $S \in C_\rho \Rightarrow S \mid \mathscr{H}_0 \in C_\rho$. Applying this comment to $(w_\rho(T))^{-1} T$, which has each $\mathscr{H}_n$ as an invariant subspace, we see that $(w_\rho(T))^{-1} T_n = (w_\rho(T))^{-1} T \mid \mathscr{H}_n \in C_\rho$, so that $w_\rho(T_n) \leq w_\rho(T)$ for all n. It follows that $a \leq w_\rho(T)$. ∎

THEOREM 4.2. *If* $T_n \in B(\mathscr{H}_n)$ $(n = 1, 2, \cdots)$ *and* $\sup_n \|T_n\| < \infty$, *then* $w_\infty\left(\bigoplus_1^\infty T_n\right) \geq \sup_n w_\infty(T_n)$.

Proof. Let $T = \bigoplus_1^\infty T_n$. For each n, there exists an approximate eigenvalue λ of T_n such that $|\lambda| = \nu(T_n)$ $(= w_\infty(T_n))$. Since T_n is the restriction of T to an invariant subspace, λ is also an approximate eigenvalue of T. Hence $w_\infty(T)$ $(= \nu(T)) \geq |\lambda| = w_\infty(T_n)$. ∎

Note that inequality can occur in Theorem 4.2. For example, let $\mathscr{H}_n$ be an n-dimensional space with orthonormal basis $\phi_1, \phi_2, \cdots, \phi_n$, and let $T_n \phi_k = \phi_{k+1}$ $(1 \leq k \leq n - 1)$, and $T_n \phi_n = 0$. Clearly, $T_n^n = 0$, so that $\sup_n w_\infty(T_n) = 0$. However, $\left\|\left(\bigoplus_1^\infty T_n\right)^k\right\| = \sup_n \|T_n^k\| = 1$, for all k. By the spectral radius formula, $w_\infty\left(\bigoplus_1^\infty T_n\right) = 1$.

The next theorem shows that under appropriate restrictions on the dimensions of the spaces $\mathscr{H}_n$, such behavior is impossible.

THEOREM 4.3. *Suppose that* $T_n \in B(\mathscr{H}_n)$ $(n = 1, 2, \cdots)$, *that* $\sup_n \|T_n\| < \infty$, *and that* $\sup_n (\dim(\mathscr{H}_n)) < \infty$. *Then* $w_\infty\left(\bigoplus_1^\infty T_n\right) = \sup_n w_\infty(T_n)$.

Proof. Let $T = \bigoplus_1^\infty T_n$ and $\mathscr{H} = \bigoplus_1^\infty \mathscr{H}_n$. In view of Theorem 4.2, we need only show that $\nu(T) \leq \sup_n \nu(T_n)$. But, if $|\lambda| > \sup_n \nu(T_n)$, then, for each n, the operator $\lambda I - T_n$ has an inverse $(\lambda I - T_n)^{-1} \in B(\mathscr{H}_n)$. Clearly, $\bigoplus_1^\infty (\lambda I - T_n)^{-1}$ is an inverse for $\lambda I - T$ on $\mathscr{H}$, provided it is well-defined, that is, provided $\sup_n \|(\lambda I - T_n)^{-1}\| < \infty$. If this is not the case, however, we may assume (by passing to a subsequence) that there exist $h_n \in \mathscr{H}_n$ with $\|h_n\| = 1$ and $\|(\lambda I - T_n)h_n\| \to_n 0$. Since $\sup_n \dim(\mathscr{H}_n) < \infty$, we may also assume that all the $\mathscr{H}_n$ are of the same dimension. In fact, we may identify each $\mathscr{H}_n$ with a single finite-dimensional space $\mathscr{H}_0$. Now $\sup_n \|T_n\| < \infty$ and $T_n \in B(\mathscr{H}_0)$ for each n; therefore, by compactness, we may assume that $T_n \to_n T_0$ (convergence in operator norm), for some $T_0 \in B(\mathscr{H}_0)$. Now

$$\|(\lambda I - T_0)h_n\| \leq \|(\lambda I - T_n)h_n\| + \|T_n - T_0\| \, \|h_n\| \to_n 0,$$

so that $\lambda \in \sigma(T_0)$. But then $\nu(T_0) > \sup_n \nu(T_n)$, which is impossible, since $T_n \to T_0$ and it is well-known that the spectrum is continuous in the operator norm if the underlying space is of finite dimension. ∎

THEOREM 4.4. *Suppose that* $0 < \rho_1 < \rho < \rho_2 < \infty$ *and that* $0 < a < 1$. *Then there exists some* $b < 1$ *such that* $w_\rho(T) \leq b$ *whenever* T *is an operator satisfying the inequalities* $w_{\rho_1}(T) \leq 1$ *and* $w_{\rho_2}(T) \leq a$.

Proof. Let $\mathscr{A}$ be the class of operators T with $w_{\rho_1}(T) \leq 1$ and $w_{\rho_2}(T) \leq a$. Let $b = \sup \{w_\rho(T): T \in \mathscr{A}\}$. Choose $T_n \in \mathscr{A}$ so that $w_\rho(T_n) \uparrow_n b$. Note that $\sup_n \|T_n\| \leq (2\rho_1 - 1) \vee 1$, by (2.2), since $w_{\rho_1}(T_n) \leq 1$. Thus we may construct the operator $T = \bigoplus_1^\infty T_n$, and it follows from Theorem 4.1 that $w_\rho(T) = b$, $w_{\rho_2}(T) \leq a$, and $w_{\rho_1}(T) \leq 1$. But, by (2.2), $w_{\rho_1}(T) \geq w_\rho(T)$, so that if $b = 1$, then $w_{\rho_1}(T) = w_\rho(T) = 1$. But in this case, by (2.12), $w_{\rho_2}(T) = 1$. This is not possible, since $a < 1$, and we conclude that $b < 1$. ∎

THEOREM 4.5. *Suppose that* $0 < \rho_1 < \rho < \infty$, *that* N *is an integer, and that* $0 \leq a < 1$. *Then there exists some* $b < 1$ *such that* $w_\rho(T) \leq b$ *whenever* T *is an operator on a space of dimension at most* N *and* T *satisfies the inequalities* $w_{\rho_1}(T) \leq 1$ *and* $w_\infty(T) \leq a$.

Proof. We simply repeat the proof of Theorem 4.4 with $\rho_2 = \infty$, except that we appeal to Theorem 4.3 at the obvious point (this requires the hypothesis involving N). ∎

If T is any operator such that $T^2 = 0$, then $w_\rho(T) = \rho^{-1} \|T\|$ for all ρ (this is the assertion (2.11)). We do not have an exact expression for $w_\rho(T)$ when we know only that $T^k = 0$ for some integer $k > 2$. However, the next result says something about the rate at which $w_\rho(T)$ converges to 0 (= $\nu(T)$) in this case.

THEOREM 4.6. *For each integer* k, *there exists a nonincreasing function* $F_k(\rho)$ *on* $(0, \infty)$ *such that* $\lim_{\rho\to\infty} F_k(\rho) = 0$ *and* $F_k(\rho)$ *satisfies the relation* $w_\rho(T) \leq F_k(\rho) \|T\|$ *whenever* T *is an operator such that* $T^k = 0$. *On the other hand, if* T *is such an operator, then* $w_\rho(T) \geq \rho^{-(1/j)} \|T^j\|^{(1/j)}$, *where* j *is the smallest integer such that* $2j \geq k$.

Proof. Let $\mathscr{A}$ be the class of operators T such that $\|T\| = 1$ and $T^k = 0$, and set $F_k(\rho) = \sup\{w_\rho(T): T \in \mathscr{A}\}$ $(0 < \rho < \infty)$. Since each $w_\rho(T)$ is nonincreasing in ρ, $F_k(\rho)$ also has this property. We have only to prove that $\lim_{\rho\to\infty} F_k(\rho) = 0$. Otherwise, $w_{\rho_n}(T_n) \geq \varepsilon > 0$ for some sequence $\{\rho_n\}$ such that $\rho_n \uparrow \infty$ and certain operators $T_n \in \mathscr{A}$. Let $T = \bigoplus_1^\infty T_n$. By Theorem 4.1, $w_\rho(T) \geq \varepsilon$ for all ρ $(0 < \rho < \infty)$. Hence, by (2.1), $\nu(T) \geq \varepsilon$. But this is impossible, since $T^k = \bigoplus_1^\infty T_n^k = 0$.

To prove the second half of the theorem, note that $(T^j)^2 = 0$. By (2.11), $w_\rho(T^j) = \rho^{-1} \|T^j\|$. But (2.7) implies that $w_\rho(T^j) \leq (w_\rho(T))^j$. ■

5. MULTIPLICATIVE INEQUALITIES FOR $w_\rho(\cdot)$

We now apply the results and techniques of the last section to the problem of establishing inequalities of the form $w_{\rho\sigma}(TS) \leq K\, w_\rho(T)\, w_\sigma(S)$, under the hypothesis that T and S are commuting operators. This seems a rather intractable problem, although (2.8) asserts that is has a neat solution $(K = 1)$ when T and S are doubly commuting operators. Here we shall study only the case where $\sigma = 1$, so that we seek inequalities of the form $w_\rho(TS) \leq K\, w_\rho(T) \|S\|$. Of course, regardless of whether T and S commute, if $\rho > 1$, then

$$w_\rho(TS) \leq \|TS\| \leq \|T\| \, \|S\| \leq \rho\, w_\rho(T) \|S\|$$

(in the last step, we have used (2.3)). In the following theorems, we improve this result by appealing to the inequality $\nu(TS) \leq \nu(T)\,\nu(S)$, which *does* depend on commutativity.

THEOREM 5.1. *For each* $\rho > 1$, *there is a constant* $K(\rho) < \rho$ *such that* $w_\rho(TS) \leq K(\rho)\, w_\rho(T) \|S\|$ *whenever* T *and* S *are commuting operators.*

Proof. Otherwise, we have commuting pairs (T_n, S_n), normalized so that $\|T_n\| = \|S_n\| = 1$, and such that $(w_\rho(T_n))^{-1} w_\rho(T_n S_n) = r_n \uparrow_n \rho$. But then $w_\rho(T_n S_n) = w_\rho(T_n)\, r_n \geq r_n \rho^{-1}$, by (2.3). Let $T = \bigoplus_1^\infty T_n$ and $S = \bigoplus_1^\infty S_n$. Then $TS = ST = \bigoplus_1^\infty T_n S_n$, and hence $w_\rho(TS) = \sup_n w_\rho(T_n S_n) \geq \sup_n r_n \rho^{-1} = 1$. But $w_\rho(TS) \leq \|TS\| \leq \|T\| \, \|S\| = 1$, so that, by (2.12), $\nu(TS) = 1$. It follows that

$$1 \leq \nu(T)\,\nu(S) \leq w_\rho(T)\,\|S\| = w_\rho(T) = \sup_n w_\rho(T_n)\,.$$

Since $w_\rho(T_n S_n) = r_n w_\rho(T_n)$ and $r_n \uparrow_n \rho$, we conclude that $\sup_n w_\rho(T_n S_n) = \rho > 1$. This is not possible, because $w_\rho(T_n S_n) \leq \|T_n S_n\| \leq 1$. ■

We now turn to a result of this type that depends on the constants provided by Theorem 4.5. Suppose that N is an integer and that $1 < \rho < \infty$. It follows from Theorem 4.5 that, for some $b < 1$, $w_\rho(A) \leq b$ whenever $A \in B(\mathscr{H})$, $\dim(\mathscr{H}) \leq N$, $\|A\| \leq 1$, and $\nu(A) \leq \rho^{-1/2}$.

THEOREM 5.2. *Suppose that* N, ρ, *and* b *are related as in the last paragraph. Then* $w_\rho(TS) \leq b\rho\, w_\rho(T)\,\|S\|$ *whenever* T, $S \in B(\mathscr{H})$, $\dim(\mathscr{H}) \leq N$, *and* T *and* S *commute.*

Proof. Suppose that $w_\rho(TS) > b\rho\, w_\rho(T)\,\|S\|$. Then, by (2.3),

$$w_\rho(TS) > b\,\|T\|\,\|S\| \geq b\,\|TS\|\,,$$

so that $\nu(TS/\|TS\|) > \rho^{-1/2}$. It follows that

$$b\rho\, w_\rho(T)\,\|S\| < w_\rho(TS) \leq \|TS\| < \rho^{1/2}\nu(TS) \leq \rho^{1/2}\nu(T)\,\nu(S) \leq \rho^{1/2} w_\rho(T)\,\|S\|\,,$$

so that $b < \rho^{-1/2}$. This is impossible (in the notation of the paragraph introducing this theorem, put $A = \rho^{-1/2} I$). ■

Our last theorem shows that if we again restrict the dimension of the underlying space, we can reduce the constant $K(\rho)$ to an arbitrarily small multiple of ρ, provided ρ is large enough.

THEOREM 5.3. *For each integer* N *and each* $\varepsilon > 0$, *there exists* $\rho_0 < \infty$ *such that* $w_\rho(TS) \leq \varepsilon\rho\, w_\rho(T)\,\|S\|$ *whenever* T, $S \in B(\mathscr{H})$, $\dim \mathscr{H} \leq N$, $TS = ST$, *and* $\rho \geq \rho_0$.

Proof. Otherwise, we have a sequence $\{\rho_n\}$ $(\rho_n \uparrow \infty)$ and commuting pairs (T_n, S_n) such that $T_n, S_n \in B(\mathscr{H})$, $\|T_n\| = \|S_n\| = 1$, and

$$w_{\rho_n}(T_n S_n) > \varepsilon\, \rho_n\, w_{\rho_n}(T_n)\,\|S_n\|\,.$$

By (2.3), $w_{\rho_n}(T_n S_n) > \varepsilon\,\|T_n\|\,\|S_n\| = \varepsilon$. Let $T = \bigoplus_1^\infty T_n$ and $S = \bigoplus_1^\infty S_n$. Then $TS = ST = \bigoplus_1^\infty T_n S_n$, and, by Theorem 4.1, $w_{\rho_n}(TS) \geq w_{\rho_n}(T_n S_n) > \varepsilon$. From (2.1) it follows that $\nu(TS) \geq \varepsilon$, so that

$$\varepsilon \leq \nu(T)\,\nu(S) \leq \nu(T)\,\|S\| = \nu(T) = \sup_n \nu(T_n)$$

(at the last step, we invoke Theorem 4.3). Now

$$\nu(T_n) \leq w_{\rho_n}(T_n) < (\varepsilon\rho_n\,\|S_n\|)^{-1}\, w_{\rho_n}(T_n S_n) \leq (\varepsilon\rho_n)^{-1}\,.$$

But we can assume, if we wish, that $\rho_n \geq \varepsilon^{-2}$ for all n, and a contradiction results. ■

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