SOME INTRICATE NONINVERTIBLE LINKS

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Let L be an oriented, ordered link tamely imbedded in the oriented 3-sphere S^3, and let μ and κ be integers such that $1 \leq \kappa < \mu$. We shall say that L is a *generalized noninvertible link with respect to the pair* (μ, κ) (or a (μ, κ)I-*link*) if it satisfies the following three conditions:

(i) L *has* μ *components;*

(ii) *each sublink with* κ *or fewer components is invertible;*

(iii) *each sublink with more than* κ *components is noninvertible.*

L is *invertible* provided it is of the same (oriented) type as its inverse. The *inverse* differs from L only in the orientation of each component.

Now (2, 1)I-links were exhibited in [6], and a $(\mu, \mu - 1)$I-link was given in [7] for each $\mu \geq 3$ (see also Figure 1). In this paper, we complete the picture by constructing a generalized noninvertible link for each pair (μ, κ) such that $1 \leq \kappa < \mu$ and $\mu \geq 3$. As an example, a (4, 2)I-link is given in Figure 4.

1. TWO PROPOSITIONS

The following two propositions clear the way for the constructive proof of our theorem in Section 3.

PROPOSITION 1. *For each integer* $\mu \geq 2$, *there exists a* $(\mu, 1)$I-*link in* S^3.

Proof. Each component of each (2, 1)I-link of [6] is of knot type 5_1. As an induction hypothesis, suppose that L is a $(\mu, 1)$I-link with $\mu \geq 2$ and that each component of L is of knot type 5_1. Let $K_{\mu+1}$ denote an oriented knot of type 5_1 in $S^3 - L$, and suppose that for each $\nu = 1, \cdots, \mu$ it represents an element of $\pi_1(S^3 - K_\nu)$ that cannot be mapped onto its inverse by any inversion [5] of this group. By [6], such an element of $\pi_1(S^3 - K_\nu)$ exists. In conjunction with the induction hypothesis, this means that each sublink of $L \cup K_{\mu+1}$ of two or more components is noninvertible. Hence, $L \cup K_{\mu+1}$ is a $(\mu + 1, 1)$I-link, and the conclusion follows by induction.

PROPOSITION 2. *For each integer* $\mu \geq 2$, *there exists a* $(\mu, \mu - 1)$I-*link in* S^3.

Proof. This proposition states the combined contents of [6] and [7]. However, in view of our objective in this paper of constructing generalized noninvertible links, it is convenient to give for each $\mu \geq 3$ a $(\mu, \mu - 1)$I-link different from that described in [7].

The link L of Figure 1 is assumed to have $\mu \geq 3$ components, each of which is of trivial knot type. Note that the sublink $L^* = K_2 \cup \cdots \cup K_\mu$ is a link of Brunnian type [1], so that L^* is unsplittable while each of its proper sublinks is completely splittable. Furthermore, it is easy to see that each proper sublink of L is invertible. A proof that L is noninvertible can be constructed along the lines given in [7].

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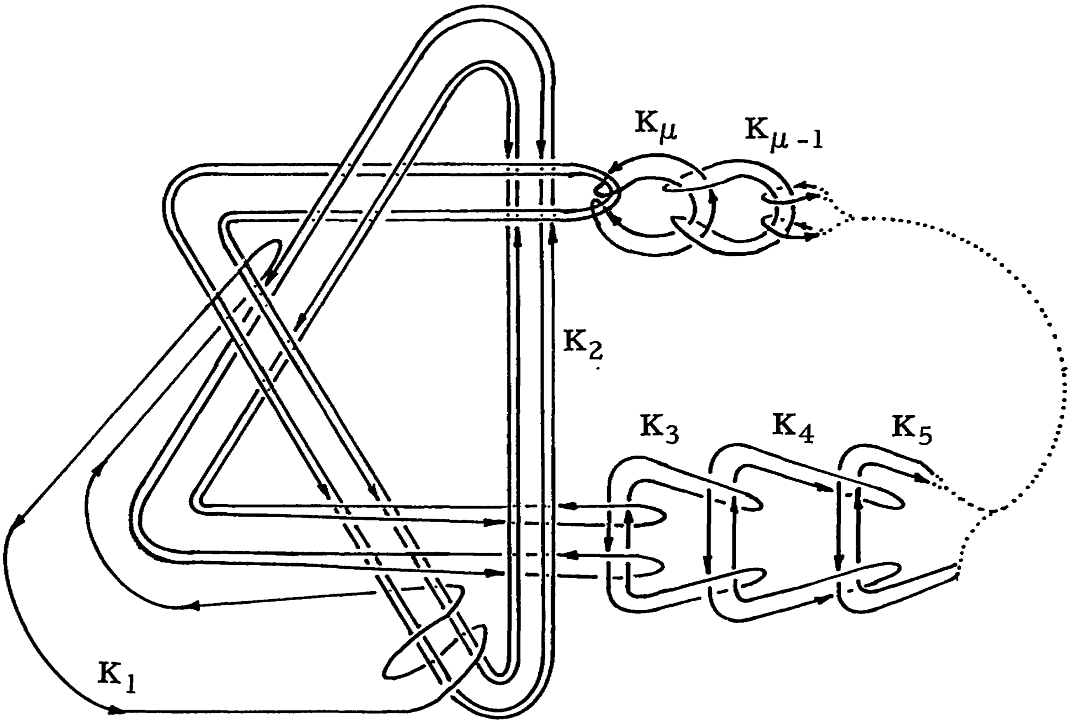

Figure 1.

Instead of giving all details, we content ourselves with an outline of the steps involved.

2. NONINVERTIBILITY OF L

Let V denote a closed solid torus tamely imbedded in $S^3 - K_1$. Let the core of V be of knot type 5_1, and assume that Int V contains the sublink $L^* = K_2 \cup \cdots \cup K_\mu$ of L in the obvious nice way.

Step 1. The first step is the establishment of a theorem to the effect that

$$\pi_1(S^3 - L^*) \approx \pi_1(S^3 - \text{Int } V) \underset{\pi_1(\partial V)}{*} \pi_1(V - L^*) .$$

Step 2. Let Σ denote the unbranched covering space of $S^3 - L^*$ corresponding to $\pi_1(S^3 - \text{Int } V)$, and assume that ψ is an inversion of L in S^3. Lift ψ to an automorphism ϕ of $\pi_1(\Sigma)$ that takes a certain element v of $\pi_1(\Sigma)$ onto its inverse. Under the projection isomorphism, v corresponds to an element w of $\pi_1(S^3 - \text{Int } V)$ represented by the (oriented) component K_1 considered as a loop.

Step 3. Finally, we must show that no such automorphism ϕ of $\pi_1(\Sigma)$ exists. This is possible because of the particularly simple form of the automorphisms of torus knot groups [4], and because in such groups the conjugacy problem can be solved modulo the center (see Theorem 1.4, p. 40 of [3]).

Remark. The proof of noninvertibility as outlined here represents a slight simplification over that in [7]. The difference lies in Steps 1 and 2.

3. (μ, κ)I-LINKS

THEOREM. *For each pair of integers* μ *and* κ $(1 \leq \kappa < \mu)$, *there is a generalized noninvertible link* $\mathscr{L}$ *in* S^3 *satisfying conditions* (i), (ii), *and* (iii) *of the Introduction.*

Proof. By Propositions 1 and 2, we need consider only those pairs (μ, κ) for which $2 \leq \kappa < \mu - 1$. We relax this, however, and assume only that $2 \leq \kappa < \mu$. Our proof is constructive.

Let $C^\lambda = \{\alpha_{\lambda 2}, \cdots, \alpha_{\lambda,\kappa+1}\}$ $(\lambda = 1, \cdots, \nu)$ denote the distinct combinations of the integers $2, \cdots, \mu$ taken κ at a time. To be definite, let us say that C^λ is the λth combination in the lexicographical ordering of the combinations. For $\lambda = 1, \cdots, \nu$, define

$$\mathfrak{C}^\lambda = \{1, 2, \cdots, \alpha_{\lambda 2} - 1, \alpha_{\lambda 2}, \cdots, \alpha_{\lambda,\kappa+1}\} .$$

We assign to each $\mathfrak{C}^\lambda$ a link in S^3 as follows. Let Q_λ $(\lambda = 1, \cdots, \nu)$ denote a collection of disjoint (tame) 3-cells in S^3. In fact, we shall assume that each Q_λ is in the shape of a solid cylinder. For $\lambda = 1, \cdots, \nu$, construct the oriented, ordered link

$$L_\lambda = K_1^\lambda \cup \cdots \cup K^\lambda_{\alpha_{\lambda 2}-1} \cup K_{\lambda 2} \cup \cdots \cup K_{\lambda,\kappa+1}$$

in Q_λ as shown in Figure 2. All of L_λ except two small arcs of each component is to lie in Int Q_λ; the exceptional arcs are to lie on ∂Q_λ as indicated. L_λ has the following properties:

1. For $j = 1, \cdots, \alpha_{\lambda 2} - 1$, the sublink $K_j^\lambda \cup K_{\lambda 2} \cup \cdots \cup K_{\lambda,\kappa+1}$ is of the same (oriented) type as the link L of Figure 1 taken with $\kappa + 1$ components.

2. Any sublink of L_λ obtained by removal of one of the components $K_{\lambda 2}, \cdots, K_{\lambda,\kappa+1}$ is completely splittable.

In the final phase of the construction, the components of the L_λ are composed among themselves. In order to describe how the compositions are to be formed, it is convenient to change the name of each component to a pair of numbers. In L_λ, denote K_j^λ by (λ, j) for $j = 1, \cdots, \alpha_{\lambda 2} - 1$, and $K_{\lambda k}$ by $(\lambda, \alpha_{\lambda k})$ for $k = 2, \cdots, \kappa + 1$.

Now, for $\alpha = 1, \cdots, \mu$, let $(\lambda_1, \alpha), \cdots, (\lambda_{t(\alpha)}, \alpha)$ be the collection of all those pairs whose second coordinate is α. Assume that $\lambda_1 < \cdots < \lambda_{t(\alpha)}$. Now set

$$\mathscr{K}_\alpha = (\lambda_1, \alpha) \# \cdots \# (\lambda_{t(\alpha)}, \alpha) .$$

(For a complete and interesting account of the composition operation #, see R. H. Fox [2, Section 7, p. 139].) The compositions, formed inductively with respect to α, are made by running two parallel arcs (each with proper orientation) from (λ_m, α) to (λ_{m+1}, α) $(m = 1, \cdots, t(\alpha) - 1)$, as indicated in Figure 3; then delete the appropriate small arc on ∂Q_{λ_m} as well as the one on $\partial Q_{\lambda_{m+1}}$. The two arcs used in composing (λ_m, α) with (λ_{m+1}, α) are to lie in a small, closed, tubular neighborhood $N(m, \alpha)$ of one of the arcs. The closed tubular neighborhoods $N(m, \alpha)$ $(m = 1, \cdots, t(\alpha) - 1;\ \alpha = 1, \cdots, \mu)$ are to be pairwise disjoint, and each $N(m, \alpha)$ meets the two 3-cells Q_{λ_m} and $Q_{\lambda_{m+1}}$ and no others. Furthermore, the intersection of $N(m, \alpha)$ with each of Q_{λ_m} and $Q_{\lambda_{m+1}}$ is to be a small disk on the boundary of each cell. Finally, set $\mathscr{L} = \mathscr{K}_1 \cup \cdots \cup \mathscr{K}_\mu$.

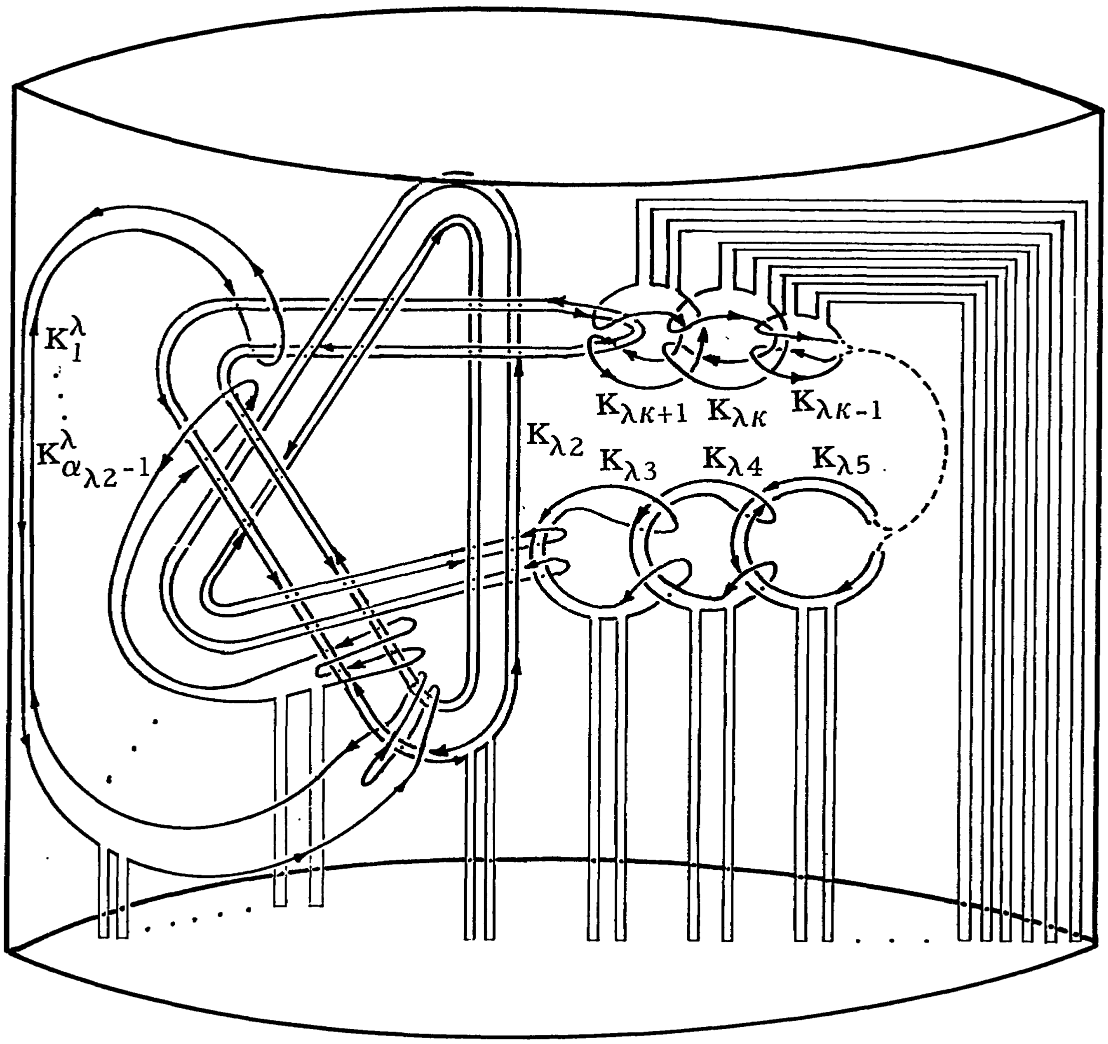

Figure 2.

It is evident from the definition of the sets $\mathfrak{C}^\lambda$ that $\mathscr{L}$ has exactly μ components, and hence, satisfies (i). To see that $\mathscr{L}$ satisfies (ii), let $\mathscr{K}_{\alpha_1} \cup \cdots \cup \mathscr{K}_{\alpha_\kappa}$ be any sublink of $\mathscr{L}$ with κ components. (Since the property of invertibility or non-invertibility of a link is independent of the order of the components of the link, we may assume that $\alpha_1 < \cdots < \alpha_\kappa$.) We have two cases: either $\{\alpha_1, \cdots, \alpha_\kappa\}$ is one of the $\nu = \binom{\mu - 1}{\kappa}$ combinations $C^\lambda = \{\alpha_{\lambda 2}, \cdots, \alpha_{\lambda,\kappa+1}\}$, or it is not.

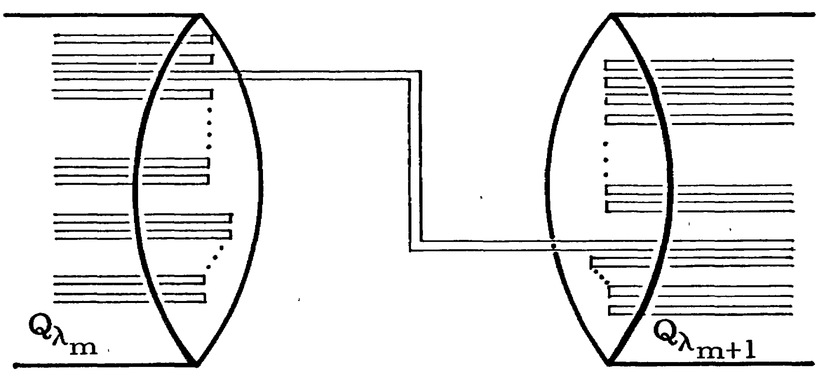

Figure 3.

In the first case, $\{\alpha_1, \cdots, \alpha_\kappa\} = \{\alpha_{\lambda 2}, \cdots, \alpha_{\lambda,\kappa+1}\}$ for some λ, and the link $\mathscr{K}_{\alpha_1} \cup \cdots \cup \mathscr{K}_{\alpha_\kappa}$ is of the same (oriented) type as the sublink

$$L_\lambda^* = K_{\lambda 2} \cup \cdots \cup K_{\lambda,\kappa+1}$$

of L_λ. In fact, we can arrange an isotopic deformation of S^3 that takes $\mathscr{K}_{\alpha_1} \cup \cdots \cup \mathscr{K}_{\alpha_\kappa}$ onto L_λ^*, as may be seen by studying Figure 2. (In this connection, note property 2 above of L_λ.) In the second case, $\mathscr{K}_{\alpha_1} \cup \cdots \cup \mathscr{K}_{\alpha_\kappa}$ is completely splittable, as may again be seen by considering Figure 2. In the first case, the invertibility of $\mathscr{K}_{\alpha_1} \cup \cdots \cup \mathscr{K}_{\alpha_\kappa}$ follows from the fact that L_λ^* is invertible (see property 1 above of L_λ). In the second case, the invertibility is obvious. Thus, $\mathscr{L}$ satisfies (ii).

To see that $\mathscr{L}$ satisfies (iii), let $\mathscr{K}_{\alpha_1} \cup \cdots \cup \mathscr{K}_{\alpha_{\kappa+1}}$ be any sublink of $\mathscr{L}$ with $\kappa + 1$ components. Then $\{\alpha_2, \cdots, \alpha_{\kappa+1}\} = \{\alpha_{\lambda 2}, \cdots, \alpha_{\lambda,\kappa+1}\}$ for some $\lambda = 1, \cdots, \nu$, so that $\mathscr{K}_{\alpha_2} \cup \cdots \cup \mathscr{K}_{\alpha_{\kappa+1}}$ is of the same (oriented) type as L_λ^*. (We assume that $\alpha_1 < \cdots < \alpha_{\kappa+1}$.) By Step 1 of Section 2, it follows that

$$\pi_1(S^3 - \mathscr{K}_{\alpha_2} \cup \cdots \cup \mathscr{K}_{\alpha_{\kappa+1}}) \approx \pi_1(S^3 - \operatorname{Int} V) \underset{\pi_1(\partial V)}{*} \pi_1(V - \mathscr{K}_{\alpha_2} \cup \cdots \cup \mathscr{K}_{\alpha_{\kappa+1}}),$$

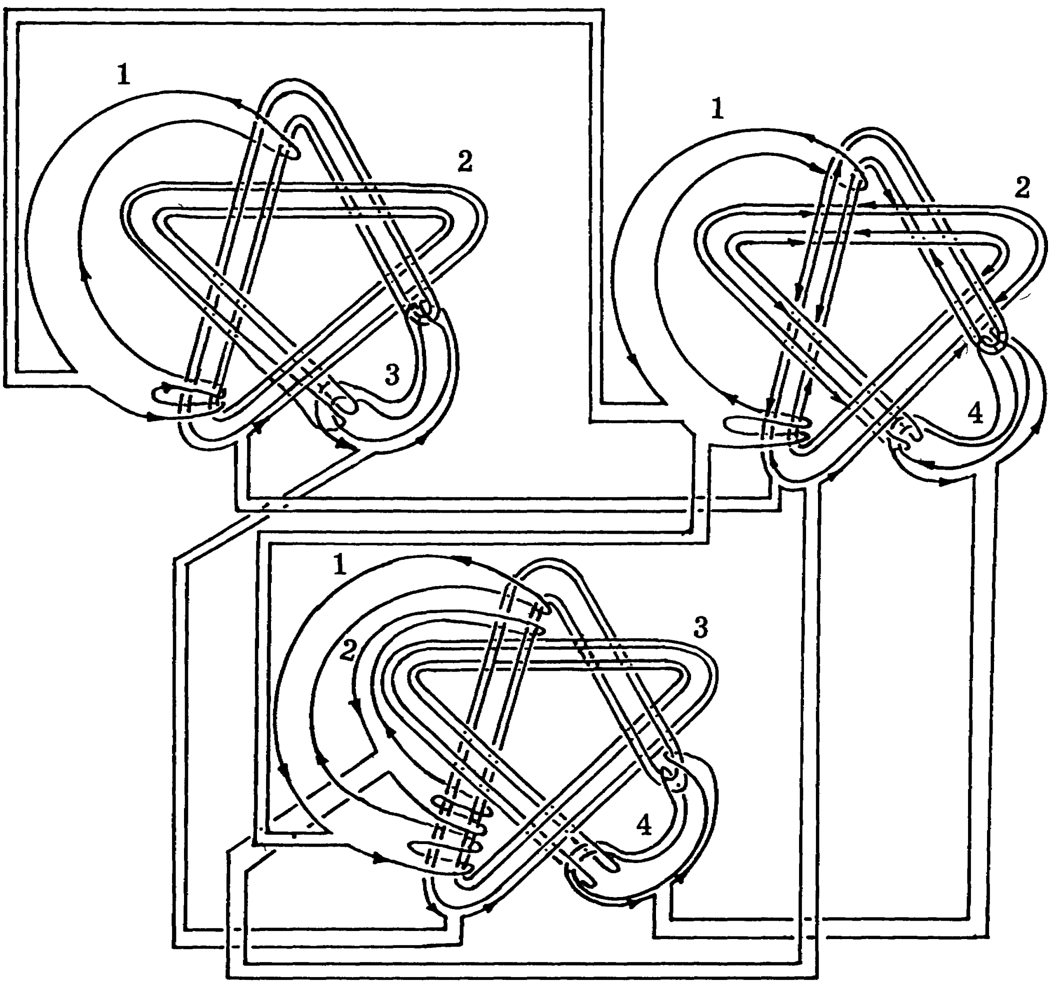

Figure 4.

where V is an appropriate solid torus. To prove the noninvertibility of $\mathscr{K}_{\alpha_1} \cup \cdots \cup \mathscr{K}_{\alpha_{\kappa+1}}$, we now follow Steps 2 and 3 of Section 2. The only point that should be noted here is that the (oriented) component $\mathscr{K}_{\alpha_1}$, considered as a loop, represents an element of $\pi_1(S^3 - \text{Int } V)$. This is easy to see by referring to Figure 2 and considering the construction. This completes the proof of the theorem.

In conclusion, we illustrate in Figure 4 one of the (μ, κ)I-links constructed in the above theorem; it covers the simplest case: $\mu = 4$, $\kappa = 2$.

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