SOME INTERPOLATION PROBLEMS IN HILBERT SPACES

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INTRODUCTION

In 1961, H. S. Shapiro and A. L. Shields [9] investigated interpolation problems in several function spaces. The present paper is an extension of the part of their work that treats weighted interpolation (by pointwise evaluation at a sequence of points) in several classical Hilbert spaces, especially H_2. First we shall obtain results concerning interpolation by sequences of arbitrary continuous linear functionals in an arbitrary Hilbert space, and later we shall obtain more specialized results involving interpolation by evaluation of derivatives in classical Hilbert spaces.

Let $\{z_i\}$ denote a sequence of points in the disk $D = [|z| < 1]$. Then $\{z_i\}$ is called a *Carleson sequence* if

$$\prod_{i \neq j} |(z_i - z_j)/(1 - \bar{z}_j z_i)| \geq \delta > 0 \qquad (j = 1, 2, \cdots).$$

The sequence $\{z_i\}$ is called an *exponential sequence* if

$$(1 - |z_{j+1}|)/(1 - |z_j|) \leq r < 1 \qquad (j = 1, 2, \cdots),$$

and $\{z_i\}$ is called a *radial sequence* if all the z_i lie on one radius. An exponential sequence is a Carleson sequence, and a radial Carleson sequence is an exponential sequence. Let $\mathscr{L}^1, \mathscr{L}^2, \cdots$ denote a sequence of continuous linear functionals on a Hilbert space H, let $\hat{\mathscr{L}}^i$ denote the functional $\mathscr{L}^i$ divided by its norm, and let $\hat{T}f = \{\hat{\mathscr{L}}^i f\}_{i=1}^{\infty}$. Shapiro and Shields [9] showed that if $\mathscr{L}^i$ is pointwise evaluation at z_i on the Hardy space H_2, then $\hat{T}(H_2) = \ell_2$ if and only if $\{z_i\}$ is a Carleson sequence.

In Section 2, we generalize the notions of Carleson sequence and exponential sequence and define the notion of projective sequence (which includes the radial sequences in the case of pointwise evaluation in H_2) in an arbitrary Hilbert space. We then show that if $\mathscr{L}^1, \mathscr{L}^2, \cdots$ is a sequence in the dual of the Hilbert space H, then

(i) the relation $\hat{T}(H) = \ell_2$ implies that $\{\mathscr{L}^i\}$ is a Carleson sequence;

(ii) if $\{\mathscr{L}^i\}$ is an exponential sequence, then $\hat{T}(H) \subset \ell_2$;

(iii) if $\{\mathscr{L}^i\}$ is an exponential sequence with a certain restriction, then $\hat{T}(H) \supset \ell_2$;

(iv) a projective Carleson sequence is an exponential sequence (see Theorems 2.7, 2.8, 2.9, 2.12, and Corollary 2.13).

In Note 2.14 we indicate that in general these results cannot be improved.

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J. T. Rosenbaum [5], [6] showed that if $\mathscr{L}^i$ is normalized pointwise evaluation of the nth derivative at z_i in H_2, then $\hat{T}(H_2) = \ell_2$ if $\{z_i\}$ is an exponential sequence. He showed, in fact, that this interpolation can be done simultaneously for $n = 1, 2, \cdots, N$. In Sections 3 and 4, we show that these results are true if $\{z_i\}$ is a Carleson sequence (Theorem 3.2) and that H_2 can be replaced by other classical Hilbert spaces (see Section 4). We show further that in the case of radial sequences, the condition that $\{z_i\}$ is a Carleson sequence is also necessary. We thus obtain extensions of Theorem 4 and of the corollary to Theorem 5 in [9] to higher derivatives.

The classical Hilbert spaces (besides H_2) to which we shall refer are the Bergman space A_2 of square-integrable analytic functions on D (see [2]) with kernel function $K(z, w) = (1 - z\overline{w})^{-2}$, and the space H_2' of functions that are derivatives of functions in H_2, have norm

$$\|f\|^2 = \frac{2}{\pi} \iint_D |f(z)|^2 (1 - |z|^2)\, dx\, dy < \infty,$$

and have kernel function $K(z, w) = (1 - z\overline{w})^{-3}$.

1. PRELIMINARIES

Let H denote a general Hilbert space with inner product (f, g) and norm $\|f\| = (f, f)^{1/2}$, where $f, g \in H$. Let H^0 denote the space of all bounded linear functionals on H. If $\mathscr{L} \in H^0$ has the Riesz representation $L \in H$, we denote this correspondence by $\mathscr{L} \sim L$. If $\mathscr{L}^1, \mathscr{L}^2 \in H^0$ and $\mathscr{L}^i \sim L_i$ $(i = 1, 2)$, define $(\mathscr{L}^1, \mathscr{L}^2) = (L_1, L_2)$.

Suppose further that H is a Hilbert space of functions $f(x)$ defined on a base set E and that H has a (reproducing) kernel function $K(x, y)$. We denote such a Hilbert space by HK or HK(E).

For fixed y, the kernel function $K(x, y)$ is the function providing the Riesz representation in HK of the bounded linear functional $\mathscr{L} \in HK^0$ defined by $\mathscr{L}(f) = f(y)$ for $f \in HK$. The following theorem (see for example [4, p. 318]) shows that not only pointwise evaluation but every bounded linear functional on HK has a simple representation in HK in terms of the kernel function $K(x, y)$. In fact, the representation is obtained simply by applying such a functional to $K(x, y)$ itself.

THEOREM 1.1. *If* $\mathscr{L} \in HK^0$, *then* $\mathscr{L} \sim L(x) = \overline{\mathscr{L}_y(K(y, x))}$. (Here the subscript y emphasizes that $\mathscr{L}$ operates on $K(y, x)$ in HK as a function of y.)

Proof. $L(x) = (L(y), K(y, x)) = \overline{(K(y, x), L(y))} = \overline{\mathscr{L}_y(K(y, x))}$. ■

Notation. If $\mathscr{L}^1, \mathscr{L}^2, \cdots \in H^0$, define $T: H \to \mathscr{C}^\infty$ by $Tf = \{\mathscr{L}^i f\}_{i=1}^\infty$, where $\mathscr{C}^\infty$ denotes the space of all complex sequences.

The following theorem was proved by N. Bari in [1].

THEOREM 1.2. *Let* $\mathscr{L}^i \in H^0$ *and* $\mathscr{L}^i \sim L_i$ $(i = 1, 2, \cdots)$. *If* A *is the matrix whose* (i, j)-*entry is* $[A]_{i,j} = (\mathscr{L}^i, \mathscr{L}^j) = (L_i, L_j)$, *then*

a) *the following conditions are equivalent:*

i) A *is bounded above by* M,

ii) $\sum |\mathscr{L}^i f|^2 \leq M \|f\|^2$ *for each* $f \in H$,

iii) $T(H) \subset \ell_2$;

b) *the following conditions are equivalent:*

i) A *is bounded below by* m,

ii) *for each* $c = (c_1, c_2, \cdots) \in \ell_2$, *there exists an* $f \in H$ *such that* $\mathscr{L}^i f = c_i$ *and* $m\|f\|^2 \leq \|c\|^2$,

iii) $T(H) \supset \ell_2$.

We shall need the following two results of I. Schur (see [7]).

THEOREM 1.3. *Let* (a_{ij}) *be an infinite matrix. If* $\sum_i |a_{ij}| \leq M$ *for every* j *and* $\sum_j |a_{ij}| \leq N$ *for every* i, *then* $|\sum_{i,j} a_{ij} x_i \bar{x}_j| \leq (MN)^{1/2} \sum |x_i|^2$.

THEOREM 1.4. *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be two Hermitian, positive definite* $(n \times n)$-*matrices of complex numbers. Let* α_1 *and* α_2 *be the smallest and largest eigenvalues of* A, *and let* $\beta_1 = \inf b_{ii}$, $\beta_2 = \sup b_{ii}$ $(i = 1, 2, \cdots, n)$. *Then* $C = (a_{ij} b_{ij})$ *is positive definite, and all eigenvalues of* C *lie between* $\alpha_1\beta_1$ *and* $\alpha_2\beta_2$.

Finally, we note the following well-known interpretation of the modulus of a Blaschke product.

THEOREM 1.5. *Consider points* $z_1, z_2, \cdots, z_n$ *in the unit disk* D. *Then, for* $z_0 \in D$,

$$\left|\prod_{k=1}^{n} \frac{z_0 - z_k}{1 - \bar{z}_k z_0}\right|^2 = |(a_{ij})_{i,j=0}^n| / |(a_{ij})_{i,j=1}^n|,$$

where $a_{ij} = (1 - |z_i|^2)^{1/2}(1 - |z_j|^2)^{1/2}/(1 - z_i \bar{z}_j)$.

Proof. The theorem follows if we compare formulas (2) and (3) of [8, p. 457], using $K(z, \zeta) = 1/(1 - z\bar{\zeta})$, the Szegö reproducing kernel function for H_2, and the identity

$$\left|\frac{z_k - z_j}{1 - \bar{z}_j z_k}\right|^2 = 1 - \frac{(1 - |z_j|^2)(1 - |z_k|^2)}{|1 - \bar{z}_j z_k|^2}. \blacksquare$$

Note 1.6. If $(\ ,\ )$ denotes the inner product in H_2, then the relation

$$a_{ij} = \left(\frac{(1 - |z_j|^2)^{1/2}}{1 - z\bar{z}_j}, \frac{(1 - |z_i|^2)^{1/2}}{1 - z\bar{z}_i}\right)$$

shows that (a_{ij}) is a Gram matrix of normal vectors in H_2, and hence

$$\left|\prod_{k=1}^{n} \frac{z_0 - z_k}{1 - \bar{z}_k z_0}\right|$$

is the distance of the function $\dfrac{(1 - |z_0|^2)^{1/2}}{1 - z\bar{z}_0}$ from the subspace spanned by the functions $\dfrac{(1 - |z_i|^2)^{1/2}}{1 - z\bar{z}_i}$ $(i = 1, 2, \cdots, n)$.

2. INTERPOLATION IN ARBITRARY HILBERT SPACES

Definition 2.1. If $\mathscr{L} \in H^0$, let $\hat{\mathscr{L}} = \mathscr{L}/\|\mathscr{L}\|$. We say $\hat{\mathscr{L}}$ is *$\mathscr{L}$-normalized.* If $\mathscr{L} \sim L$, then $\hat{\mathscr{L}} \sim L/\|L\| = \hat{L}$.

Definition 2.2. A sequence $\mathscr{L}^1, \mathscr{L}^2, \cdots \in H^0$ will be called a *Carleson δ-sequence* if the distance of $\hat{\mathscr{L}}^i$ to the subspace spanned by $\{\mathscr{L}^j\}_{j\neq i}$ is at least $\delta > 0$ for all i.

Definition 2.3. A sequence $\mathscr{L}^1, \mathscr{L}^2, \cdots \in H^0$ will be called an *exponential* (r, c)*-sequence* if $\|\mathscr{L}^k\|/\|\mathscr{L}^{k+1}\| \leq r < 1$ and $|(\mathscr{L}^k, \mathscr{L}^\ell)| \leq c\|\mathscr{L}^k\|^2$ for some constant c ($k, \ell = 1, 2, \cdots$).

Example 2.4. In the case of H_2, if $\mathscr{L}^i$ is pointwise evaluation at z_i ($i = 1, 2, \cdots$), then $\mathscr{L}^i \sim L_i(z) = \dfrac{1}{1 - z\bar{z}_i}$. Therefore

$$\frac{\|\mathscr{L}^k\|}{\|\mathscr{L}^{k+1}\|} = \frac{(1 - |z_{k+1}|^2)^{1/2}}{(1 - |z_k|^2)^{1/2}}$$

and

$$|(\mathscr{L}^k, \mathscr{L}^\ell)| = \frac{1}{|1 - z_\ell \bar{z}_k|} \leq \frac{1}{1 - |z_k|} \leq \frac{2}{1 - |z_k|^2} = 2\|\mathscr{L}^k\|^2 \qquad (k, \ell = 1, 2, \cdots).$$

Thus Definitions 2.2 and 2.3 are proper extensions of the notions of Carleson sequence and exponential sequence.

THEOREM 2.5. *A sequence $\mathscr{L}^1, \mathscr{L}^2, \cdots \in H^0$ is a Carleson δ-sequence if and only if there exist functions $f_k \in H$ ($k = 1, 2, \cdots$) such that $\hat{\mathscr{L}}^i f_k = \delta_{ik}$ ($i = 1, 2, \cdots$) and $\|f_k\| \leq 1/\delta$.*

Proof. To see the necessity, we take $f_k = e_k/\hat{\mathscr{L}}^k e_k$, where e_k is the Riesz representation in H of the component of $\hat{\mathscr{L}}^k$ orthogonal to the span of $\{\mathscr{L}^i\}_{i\neq k}$. For the sufficiency, observe that

$$\frac{1}{\delta}\Big\|\hat{\mathscr{L}}^k - \sum_{i\neq k} c_i\mathscr{L}^i\Big\| \geq \Big|\Big(\hat{\mathscr{L}}^k - \sum_{i\neq k} c_i\mathscr{L}^i\Big)(f_k)\Big| = \hat{\mathscr{L}}^k f_k = 1,$$

where Σ denotes some finite sum. ■

Definition 2.6. If $\mathscr{L}^1, \mathscr{L}^2, \cdots \in H^0$, define $\hat{T}: H \to \mathscr{C}^\infty$ by $\hat{T}f = \{\hat{\mathscr{L}}^i f\}_{i=1}^{\infty}$. (For example, if $\mathscr{L}^i$ in H_2^0 is pointwise evaluation at z_i, then

$$\hat{T}f = \{f(z_i)(1 - |z_i|^2)^{1/2}\}.)$$

If $\hat{T}(H) = \ell_2$, call $\{\mathscr{L}^i\}$ an *interpolating sequence.* If $\hat{T}(H) \supset \ell_2$, we say that normalized interpolation (or weighted interpolation, in [9], [5], and [6]) is possible.

THEOREM 2.7. *A necessary condition on $\mathscr{L}^1, \mathscr{L}^2, \cdots \in H^0$ for normalized interpolation ($\hat{T}(H) \supset \ell_2$) is that $\{\mathscr{L}^i\}$ be a Carleson sequence.*

Proof. Let $N = \{f \in H; \hat{T}f = 0\}$. Then $\hat{T}$ induces a one-to-one linear transformation $\hat{T}_*$ from H/N onto a subset of $\mathscr{C}^\infty$ containing ℓ_2. Now an application of the closed-graph theorem to $\hat{T}_*^{-1} \mid \ell_2$ shows the existence of a constant M such that

$\|f/N\| \leq M\,\|\hat{T}f\|$ for all $f \in H$, where $\|f/N\|$ denotes the quotient norm of f/N. Using Theorem 2.5, we see that $\{\mathscr{L}^i\}$ is a Carleson $1/M$-sequence. ■

THEOREM 2.8. *A sufficient condition for the inclusion* $\hat{T}(H) \subset \ell_2$ (*with bound* $M = c(1+r)/(1-r)$) *is that* $\{\mathscr{L}^i\}$ *be an exponential* (r, c)-*sequence.*

Proof. As in the proof of [9, p. 526], consider the infinite matrix (a_{ij}) with $a_{ij} = (\hat{\mathscr{L}}^i, \hat{\mathscr{L}}^j)$. Then $|a_{i(i+k)}| \leq c\|\mathscr{L}^i\|/\|\mathscr{L}^{i+k}\| \leq cr^k$. Hence, for each i,

$$\sum_{j=1}^{\infty} |a_{ij}| = \sum_{1}^{i} |a_{ij}| + \sum_{i+1}^{\infty} |a_{ij}| \leq c\sum_{1}^{i} r^{i-j} + c\sum_{i+1}^{\infty} r^{j-i}$$

$$< c\left(\frac{1}{1-r} + \frac{r}{1-r}\right) = c\,\frac{1+r}{1-r}.$$

The conclusion now follows from Theorems 1.3 and 1.2. ■

THEOREM 2.9. *A sufficient condition for the inclusion* $\hat{T}(H) \supset \ell_2$ (*with bound* $m = (1-(1+2c)r)/(1-r)$) *is that* $\{\mathscr{L}^i\}$ *be an exponential* (r, c)-*sequence, where* $r < 1/(1+2c)$.

Proof. Let $a_{ij} = (\hat{\mathscr{L}}^i, \hat{\mathscr{L}}^j)$. Then, if $\{x_i\}$ is in the unit ball of ℓ_2,

$$\left|\sum_{i,j} a_{ij} x_i \bar{x}_j\right| \geq 1 - 2\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} |a_{i(i+k)}|\,|x_i|\,|x_{i+k}|$$

$$\geq 1 - 2\sum_{k=1}^{\infty} cr^k \sum_{i=1}^{\infty} |x_i|\,|x_{i+k}| \geq \frac{1-(1+2c)r}{1-r},$$

since $|a_{i(i+k)}| \leq cr^k$. The conclusion follows from Theorem 1.2. ■

Definition 2.10. A sequence $\mathscr{L}^1, \mathscr{L}^2, \cdots \in H^0$ is a *projective* c-*sequence* if $\|\mathscr{L}^i\|^2 \leq |(\mathscr{L}^i, \mathscr{L}^{i+1})|$ and $|(\mathscr{L}^i, \mathscr{L}^j)| \leq c\,\|\mathscr{L}^i\|^2$ for some constant c $(i, j = 1, 2, \cdots)$.

Example 2.11. In the case of pointwise evaluation in H_2, $\{z_i\}$ is a radial sequence if $z_{i+1}\bar{z}_i \geq 0$ and $|z_{i+1}| > |z_i|$. Hence

$$\|\mathscr{L}^i\|^2 = \frac{1}{1-|z_i|^2} \leq \frac{1}{1-z_{i+1}\bar{z}_i} = (\mathscr{L}^i, \mathscr{L}^{i+1}).$$

Also, $(\mathscr{L}^i, \mathscr{L}^j) \leq 2\,\|\mathscr{L}^i\|^2$. Thus a radial sequence in H_2 is an example of a projective c-sequence.

THEOREM 2.12. *If* $\{\mathscr{L}^i\}$ *is a projective* c-*sequence and a Carleson* δ-*sequence, it is an exponential* $(\sqrt{1-\delta^2}, c)$-*sequence.*

Proof. If $\{\mathscr{L}^i\}$ is a projective Carleson δ-sequence, then

$$\sqrt{1-\delta^2} \geq |(\hat{\mathscr{L}}^i, \hat{\mathscr{L}}^{i+1})| = \frac{|(\mathscr{L}^i, \mathscr{L}^{i+1})|}{\|\mathscr{L}^i\|\,\|\mathscr{L}^{i+1}\|} \geq \frac{\|\mathscr{L}^i\|}{\|\mathscr{L}^{i+1}\|},$$

and we have an exponential sequence. ■

COROLLARY 2.13. *A necessary and sufficient condition that a projective* c-*sequence* $\mathscr{L}^1, \mathscr{L}^2, \cdots \in H^0$ *be an interpolating sequence* ($\hat{T}(H) = \ell_2$) *is that* $\{\mathscr{L}^i\}$ *be a Carleson* δ-*sequence, provided that, in the case of sufficiency,*

$$\sqrt{1 - \delta^2} \leq (1 + 2c)^{-1}.$$

Note 2.14. Theorem 2.8 cannot in general be extended to Carleson sequences (as in the case of pointwise evaluation in H_2). For example, consider the sequence $\hat{L}_i = (1/\sqrt{2}, 0, \cdots, 0, 1/\sqrt{2}, 0, \cdots)$ $(i = 2, 3, \cdots)$ in $\ell_2 = H$, where the second $1/\sqrt{2}$ occurs in the ith position. The distance of $\hat{L}_i$ to the subspace spanned by the other $\hat{L}_j$ is equal to $1/\sqrt{2}$. To see this, examine the matrix (a_{ij}), where

$$a_{ij} = (\hat{L}_i, \hat{L}_j) = \begin{cases} 1 & (i = j), \\ 1/2 & (i \neq j). \end{cases}$$

We have the relation

$$\lim_{n \to \infty} \frac{|(a_{ij})_{n+1}|}{|(a_{ij})_n|} = \lim_{n \to \infty} \frac{n+1}{2n} = 1/2.$$

Thus $\{\hat{L}_i\}$ is a Carleson sequence, but the matrix (a_{ij}) is clearly not bounded above (not even finite) on ℓ_2, and hence $\hat{T}(H) \not\subset \ell_2$, by Theorem 1.2.

That Theorem 2.9 cannot be extended to exponential sequences for arbitrary $r < 1$ can be seen by the example in ℓ_2 of $L_0 = \varepsilon_1$ and $L_i = r^{-i}\varepsilon_i$ $(i = 1, 2, \cdots)$, where ε_i is the ith standard-basis element. This is an exponential sequence with $c = 1/r$, but the matrix (a_{ij}), where $a_{ij} = (\hat{L}_i, \hat{L}_j)$, has no positive lower bound, since the elements of $\{L_i\}$ are linearly dependent. Thus $\hat{T}(H) \not\supset \ell_2$, by Theorem 1.2.

It would be interesting to know whether, in general, the restriction on the sufficiency in Corollary 2.13 can be dropped.

Finally, whenever normalized interpolation is possible in the previous theorems (that is, whenever for each $c \in \ell_2$, there exists an $f \in H$ such that $\hat{T}f = c$), the interpolating function of minimum norm is given explicitly by Theorem 2.1 in [3, p. 624].

3. INTERPOLATING HIGHER DERIVATIVES IN H_2

If $z_1, z_2, \cdots \in D$, then, for a nonnegative integer n, let $\mathscr{L}^{n,i}f = f^{(n)}(z_i)$ $(i = 1, 2, \cdots)$, where $f \in H_2$. By Theorem 1.1,

$$\mathscr{L}^{n,i} \sim L_{n,i}(z) = \overline{\mathscr{L}_w^{n,i}K(w, z)} = \overline{\frac{\partial^n}{\partial w^n} K(w, z)}\Big|_{w=z_i},$$

where $K(w, z) = 1/(1 - w\bar{z})$.

LEMMA 3.1. $(\mathscr{L}^{n,j}, \mathscr{L}^{n,i}) = \sum_{k=0}^n a_k z_i^k \bar{z}_j^k/(1 - z_i\bar{z}_j)^{2n+1}$, *where all* a_k *are positive.*

Proof. The proof is by induction. If n = 0, then $(\mathscr{L}^{0,j}, \mathscr{L}^{0,i}) = \frac{1}{1 - z_i\bar{z}_j}$. Assume the statement is true for n. Then

$$(\mathscr{L}^{n+1,j}, \mathscr{L}^{n+1,i}) = \sum_{k=0}^{n+1} b_k z_i^k \bar{z}_j^k/(1 - z_i\bar{z}_j)^{2n+3} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \sum_{k=0}^{n} \frac{a_k z_i^k \bar{z}_j^k}{(1 - z_i\bar{z}_j)^{2n+1}}$$

$$= \left\{ \sum_{k=0}^{n} k^2 a_k z_i^{k-1} \bar{z}_j^{k-1} + \sum_{k=0}^{n} [(2n+1)(2k+1) - 2k^2] a_k z_i^k \bar{z}_j^k \right.$$

$$\left. + \sum_{k=0}^{n} [(2n+1)(2n+1-2k) + k^2] a_k z_i^{k+1} \bar{z}_j^{k+1} \right\} \Big/ (1 - z_i\bar{z}_j)^{2n+3} .$$

From this formula, it is evident that if each a_k is positive, then each b_k is positive. ■

THEOREM 3.2. *Let* $\hat{\mathscr{L}}^{n,i}$ *be the normalized* nth *derivative at* z_i, *that is, let*

$$\hat{\mathscr{L}}^{n,i} f = f^{(n)}(z_i) \Big/ \left[\frac{\partial^{2n}}{\partial z_i^n \partial \bar{z}_i^n} (K(z_i, z_i)) \right]^{1/2}$$

if $f \in H_2$. *If* $\{z_i\}$ *is a Carleson* δ*-sequence, then* $\hat{T}(H_2) = \ell_2$ *(with lower bound* $m = \delta^4/(2 - 4\log\delta)$ *and upper bound* $M = (2 - 4\log\delta)/\delta^4$*).*

Conversely, if $\{z_i\}$ *is a radial sequence, then the inclusion* $\hat{T}(H_2) \supset \ell_2$ *implies that it is a Carleson sequence.*

Proof. Consider the matrix (c_{ij}), where

$$c_{ij} = (\hat{\mathscr{L}}^{n,i}, \hat{\mathscr{L}}^{n,j})$$

$$= \left(\frac{(1 - |z_j|^2)^{1/2}(1 - |z_i|^2)^{1/2}}{(1 - z_i\bar{z}_j)} \right)^{2n+1} \frac{\sum_{k=0}^{n} a_k z_i^k \bar{z}_j^k}{\left(\sum_{k=0}^{n} a_k |z_i|^{2k}\right)^{1/2} \left(\sum_{k=0}^{n} a_k |z_j|^{2k}\right)^{1/2}}$$

$$= a_{ij}^{2n+1} b_{ij}$$

(here (a_{ij}) is the normalized Gram matrix corresponding to the case n = 0 investigated in [9] and found to be bounded above and below by M and m, respectively). Further, (b_{ij}) is a normalized Gram matrix of vectors $(1, z_k, z_k^2, \cdots, z_k^n) \in \mathscr{C}^{n+1}$ $(k = 1, 2, \cdots)$ with respect to the inner product $(x, y) = \sum_0^n a_k x_k \bar{y}_k$ in $\mathscr{C}^{n+1}$, by Lemma 3.1. The conclusion for the first part of the theorem then follows by 2n + 1 applications of Theorems 1.4 and 1.2.

For the converse, we have, by Theorem 2.7, Definition 2.2, and Lemma 3.1, the inequalities

$$1 > r > |(\hat{\mathscr{L}}^{n,i}, \hat{\mathscr{L}}^{n,i+1})| \geq \frac{\left(\sum a_k |z_i|^{2k}\right)^{1/2}}{\left(\sum a_k |z_{i+1}|^{2k}\right)^{1/2}} \times \left(\frac{(1 - |z_{i+1}|^2)}{(1 - |z_i|^2)} \right)^{(2n+1)/2}$$

if $\{z_i\}$ is a radial sequence. But

$$\lim_{i\to\infty}\left(\sum_{k=0}^{n} a_k |z_i|^{2k}\right)\Big/\left(\sum_{k=0}^{n} a_k |z_{i+1}|^{2k}\right) = 1 .$$

Hence, there exists an $r' < 1$ such that $(1 - |z_{i+1}|^2)/(1 - |z_i|^2) < r'$ $(i = 1, 2, \cdots)$, and thus $\{z_i\}$ is an exponential sequence. ■

Note. If $n \geq 1$, we can take $M = 1 - 2\log\delta$, by Lemma 2 of [9] and Schur's Theorem 1.3.

As a corollary, we can extend the result concerning simultaneous interpolation in [5] and [6] by combining the proof in Theorem 2 of [6] with Theorem 3.2.

COROLLARY 3.3. *If* $\{z_n\}$ *is a Carleson sequence and* M *is some nonnegative integer, then, corresponding to each choice of* $M+1$ *sequences* $w^{(0)}, \cdots, w^{(M)}$ *in* ℓ_2, *there exists an* f *in* H_2 *for which*

$$f^{(m)}(z_n) = w_n^{(m)}\left[\frac{\partial^{2m}}{\partial z_n^m \partial \bar{z}_n^m}(K(z_n, z_n))\right]^{1/2} \qquad (0 \leq m \leq M;\ n = 1, 2, \cdots) .$$

Note 3.4. Since simultaneous interpolation, as defined by Rosenbaum, is interpolation using the sequence of functionals $\mathscr{L}^{0,1}, \cdots, \mathscr{L}^{M,1}, \mathscr{L}^{0,2}, \cdots, \mathscr{L}^{M,2}, \cdots$ in the sense of Section 2, the function of minimum norm in solving the simultaneous interpolation problem can be determined explicitly by Corollary 2.2 in [3, p. 625].

4. INTERPOLATING HIGHER DERIVATIVES IN A_2 AND H_2'

The results of Section 3 carry over to analogous results in any Hilbert space of analytic functions on D whose kernel function has the form $K(w, z) = (1 - w\bar{z})^{-p}$ for some positive integer p ($p = 2$ in the case of A_2, and $p = 3$ in the case of H_2'). We maintain the same notation as in Section 3, except that now $K(w, z) = (1 - w\bar{z})^{-p}$. Then a proof similar to that for Lemma 3.1 yields the following result.

LEMMA 4.1. $(\mathscr{L}^{n,j}, \mathscr{L}^{n,i}) = \sum_{k=0}^{n} a_k z_i^k \bar{z}_j^k/(1 - z_i\bar{z}_j)^{2n+p}$, *where all* a_k *are positive.*

Note 4.2. With this lemma, one sees that Theorem 3.2 extends to the case where $K(z, w) = (1 - w\bar{z})^{-p}$, and in particular, we obtain extensions of Theorem 4 and the corollary of Theorem 5 in [9] to higher derivatives.

Also, as we can easily see by making the obvious modifications to Theorem 2 of [6] and using the above-mentioned extension of Theorem 3.2, Corollary 3.3 (simultaneous interpolation) extends to any HK(D) with $K(w, z) = (1 - w\bar{z})^{-p}$ and with the property that if $f \in HK(D)$, then $Bf \in HK(D)$, where B is a Blaschke product. Thus, in particular, Corollary 3.3 extends to A_2.

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