ON TWO INVARIANT σ-ALGEBRAS FOR AN AFFINE TRANSFORMATION

Nobuo Aoki

1. INTRODUCTION

Suppose G is a compact, connected, abelian group and T (T: G → G) is an ergodic affine transformation. We shall prove that the maximal factor transformation of T with quasi-discrete spectrum is the maximal factor of T whose entropy is zero. This result was first obtained by W. Parry [5] for the case where G is metrizable. I benefitted from reading the papers by W. Parry [5] and P. Walters [6], and I am grateful to the referee for helpful suggestions.

2. PRELIMINARIES

An affine transformation T of a compact, connected, abelian group G is a transformation of the form $T(x) = aA(x)$ $(x \in G)$, where A is a continuous group automorphism of G and where $a \in G$. Such transformations T preserve Haar measure. For a compact, connected, abelian group G with Haar measure m, we consider the normalized measure space $(G, \mathcal{E}, m)$, where $\mathcal{E}$ is the completion of the σ-algebra generated by the open subsets of G (it is not a Lebesgue space, since G is nonmetrizable).

A collection $\eta = \{E_t\}$ of $\mathcal{E}$-measurable sets with the property that

$$\bigcup_t E_t = G \quad \text{and} \quad E_t \cap E_{t'} = \emptyset \ (t \neq t')$$

is called an $\mathcal{E}$-*measurable partition.*

If ζ is an $\mathcal{E}$-measurable partition, we denote by $\mathscr{B}(\zeta)$ the σ-algebra generated by the members of ζ. Then $\mathscr{B}(\zeta)$ is a sub-σ-algebra of $\mathcal{E}$. Suppose $\{\zeta_\alpha\}$ is a collection of $\mathcal{E}$-measurable partitions. Then the algebra generated by $\bigcup_\alpha \mathscr{B}(\zeta_\alpha)$ consists of finite unions of sets of the form $\bigcap_{j=1}^n A_{\alpha_j}$, where $A_{\alpha_j} \in \mathscr{B}(\zeta_{\alpha_j})$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a finite subset of the collection of indices. By $\bigvee_\alpha \mathscr{B}(\zeta_\alpha)$, we denote the σ-algebra generated by $\bigcup_\alpha \mathscr{B}(\zeta_\alpha)$.

Suppose η is an $\mathcal{E}$-measurable partition of G; then H denotes the projection of G onto the factor space G_η; in other words, H maps a point of G onto the element of η to which it belongs. If $T\eta = \eta$ (mod 0), then the factor transformation T_η is induced by T, that is, $T_\eta = HTH^{-1}$.

Let $\mathcal{E}_\eta$ be the σ-algebra generated by the subsets of G_η that belong to the sub-σ-algebra $\mathscr{B}(\eta)$, and let m_η denote the measure on $\mathcal{E}_\eta$ induced by m. Then T_η is an automorphism of the factor space $(G_\eta, \mathcal{E}_\eta, m_\eta)$, and

Received June 30, 1969.

Michigan Math. J. 17 (1970).

$$\mathscr{B}(\eta) = H^{-1}\,\mathcal{E}_\eta\,.$$

L. M. Abramov [1] gave the definition of an automorphism of $(G, \mathcal{E}, m)$ with quasi-discrete spectrum, and F. Hahn and W. Parry [4] gave the definition of a homeomorphism with quasi-discrete spectrum. If G is a compact, connected, abelian group, then both definitions of an ergodic affine transformation with quasi-discrete spectrum on the complete normalized measure space $(G, \mathcal{E}, m)$ coincide. We use the same symbol to denote continuous group automorphisms and their duals.

If G is also metrizable and T is ergodic, then T is totally ergodic. Hence, if T is ergodic on a compact, connected, abelian group G with character group Γ, then T is totally ergodic.

Choose $f \in \Gamma$, and let $\text{gp}\{A, f\}$ denote the smallest A-invariant subgroup of Γ containing f. Then $\text{gp}\{A, f\}$ is countable. The dual space of $\text{gp}\{A, f\}$ is a compact, connected, metric, abelian group, and therefore the affine transformation induced by T is totally ergodic. We have the relation

$$\bigcup_{f\in\Gamma} \text{gp}\{A, f\} = \Gamma\,.$$

Hence, if $g \circ T^n = g$ for some integer n and some $g \in \Gamma$, then $g = 1$; that is, T is totally ergodic.

Since G is connected, the only element of finite order in the character group is the unit element. Let T be an ergodic affine transformation of G, and let

$$\Gamma_n = \{f \in \Gamma\colon B^n f = 1\}\,,$$

where $Bf(x) = (f^{-1}Af)(x)$. Then the group of quasi-proper functions of order n is $K \times \Gamma_n$ (direct product), where K is the circle group.

The spectrum of T is quasi-discrete if and only if $\bigcup_{n=1}^\infty \Gamma_n = \Gamma$. The maximal $\mathcal{E}$-measurable partition η for which the spectrum of T_η is quasi-discrete is the partition of G into cosets of $\text{ann}\left(\bigcup_{n=1}^\infty \Gamma_n\right)$, where $\text{ann}\left(\bigcup_{n=1}^\infty \Gamma_n\right)$ is the annihilator of $\bigcup_{n=1}^\infty \Gamma_n$; in other words,

$$\eta = \zeta\left(\text{ann}\left(\bigcup_{n=1}^{\infty} \Gamma_n\right)\right).$$

Such a partition is called a group partition. For the theory of entropy of measure-preserving transformations, see [3]. By $\eta(T)$, we denote the σ-algebra generated by the cosets of $\text{ann}\left(\bigcup_{n=1}^\infty \Gamma_n\right)$, and we define the σ-algebra $\pi(T)$ by the relation

$$\pi(T) = \bigvee_{\mathscr{A}\text{ finite}} \{\mathscr{A}\colon h(T, \mathscr{A}) = 0\}\,.$$

3. THE THEOREM

As before, G is a compact, connected, abelian group with character group Γ, and T ($T: G \to G$) is an ergodic affine transformation.

THEOREM. $\pi(T) = \eta(T)$, *modulo* 0.

Proof. If Λ is a subgroup of Γ, then σ-alg(ann(Λ)) denotes the σ-algebra of measurable sets that are unions of cosets of ann(Λ).

We show first that $\eta(T) \subseteq \pi(T)$. Since T_η is an affine transformation of a compact, connected, abelian group, we may suppose that $\eta(T) = \mathcal{E}$; in other words, T has quasi-discrete spectrum. If $\gamma_1, \cdots, \gamma_r \in \Gamma$, then the group

$$\mathrm{gp}\,\{A, \gamma_1, \cdots, \gamma_r\}$$

is finitely generated, and T induces a totally ergodic affine transformation with quasi-discrete spectrum on the finite-dimensional torus $G/\mathrm{ann}(\mathrm{gp}\,\{A, \gamma_1, \cdots, \gamma_r\})$. By Abramov [1], the entropy of this factor is zero. Since

$$\mathcal{E} = \bigvee_{\gamma_1, \cdots, \gamma_r} \sigma\text{-alg}(\mathrm{ann}(\mathrm{gp}\,\{A, \gamma_1, \cdots, \gamma_r\})) \pmod 0,$$

it suffices (by [3, p. 80]) to show that $h(T, \mathscr{A}) = 0$, for each finite subalgebra satisfying the relation

$$\mathscr{A} \subseteq \bigcup_{\gamma_1, \cdots, \gamma_r} \sigma\text{-alg}(\mathrm{ann}(\mathrm{gp}\,\{A, \gamma_1, \cdots, \gamma_r\})).$$

The last assertion follows from the arguments above, because there exist characters $\gamma_1, \cdots, \gamma_r$ such that

$$\mathscr{A} \subseteq \sigma\text{-alg}(\mathrm{ann}(\mathrm{gp}\,\{A, \gamma_1, \cdots, \gamma_r\})).$$

Hence $\eta(T) \subseteq \pi(T)$.

We now proceed to show the inclusion $\pi(T) \subseteq \eta(T)$ (mod 0). It suffices to show that if $\mathscr{A}$ is finite and $h(T, \mathscr{A}) = 0$, then $\mathscr{A} \subseteq \eta(T)$ (mod 0).

Let Θ denote the collection of all countable subsets of Γ. For $\alpha \in \Theta$, let $Y_\alpha = \mathrm{gp}\,\{A, \alpha\}$ (note that Y_α is a countable group). Then $G/\mathrm{ann}(Y_\alpha)$ is metrizable. If

$$T_\alpha: G/\mathrm{ann}(Y_\alpha) \to G/\mathrm{ann}(Y_\alpha)$$

is the map induced by T on $G/\mathrm{ann}(Y_\alpha)$ and if H is the projection of G onto $G/\mathrm{ann}(Y_\alpha)$, we have the inclusion

$$\sigma\text{-alg}(\mathrm{ann}(Y_\alpha)) \cap \mathscr{A} \subseteq H^{-1}(\pi(T_\alpha)).$$

But

$$\mathscr{A} = \bigvee_{\alpha \in \Theta} (\mathscr{A} \cap \sigma\text{-alg}(\mathrm{ann}(Y_\alpha))) \pmod 0.$$

One can see the last relation as follows. Choose $A \in \mathscr{A}$. Then, for each n, there exist $\alpha_n \in \Theta$ and $B_n \in \sigma\text{-alg}(\text{ann}(Y_{\alpha_n}))$ such that $m(B_n \triangle A) < 1/n$; moreover, if $\alpha = \bigcup_n \alpha_n$, then $\alpha \in \Theta$ and $A \in \sigma\text{-alg}(\text{ann}(Y_\alpha))$ (mod 0).

It follows that

$$\mathscr{A} \subseteq \bigvee_{\alpha \in \Theta} H^{-1}(\pi(T_\alpha)) \qquad (\text{mod } 0).$$

But, since $G/\text{ann}(Y_\alpha)$ is metrizable, it follows from [5] or [6] that

$$H^{-1}(\pi(T_\alpha)) \subseteq \eta(T) \qquad (\text{mod } 0),$$

and hence $\mathscr{A} \subseteq \eta(T)$ (mod 0).

REFERENCES

1. L. M. Abramov, *Metric automorphisms with quasi-discrete spectrum.* Amer. Math. Soc. Transl. (2) 39 (1964), 37-56.
2. N. Aoki, *On zero entropy and quasi-discrete spectrum for automorphisms.* Proc. Japan Acad. 45 (1969), 20-24.
3. P. Billingsley, *Ergodic theory and information.* John Wiley and Sons, Inc., New York, 1965.
4. F. Hahn and W. Parry, *Minimal dynamical systems with quasi-discrete spectrum.* J. London Math. Soc. 40 (1965), 309-323.
5. W. Parry, *On the coincidence of three invariant σ-algebras associated with an affine transformation.* Proc. Amer. Math. Soc. 17 (1966), 1297-1302.
6. P. Walters, *On the relationship between zero entropy and quasi-discrete spectrum for affine transformations.* Proc. Amer. Math. Soc. 18 (1967), 661-667.

Josai University
Sakado, Saitama
Japan