LINDELÖF REALCOMPACTIFICATIONS

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A topological space X is called an I-space if every collection of closed sets with the countable intersection property (c.i.p.) is contained in a maximal collection of closed sets with the c.i.p. This notion was introduced by R. W. Bagley and J. D. McKnight [1]. Examples of I-spaces are the Lindelöf spaces and the countably compact spaces. In this note, we examine under what conditions the realcompactification υX of an I-space X is a Lindelöf space. We also settle a question raised by the paper of Bagley and McKnight.

We refer the reader to L. Gillman and M. Jerison [3] for such matters as the definition and the basic properties of υX, where X is a completely regular space, and for terminology. For example, a z-filter is a "filter" of zero sets of continuous, real-valued functions on X [3, page 24]. All spaces in this paper are completely regular.

LEMMA 1. *The realcompactification* υX *of* X *is a Lindelöf space if and only if every* z*-filter in* X *with the* c.i.p. *is contained in a* z*-ultrafilter with the* c.i.p.

Proof. Note that if Z is the zero set of a continuous real function f on X and $\mathrm{cl}_{\upsilon X}$ denotes the closure operator in υX, then $\mathrm{cl}_{\upsilon X} Z$ is the zero set of f^{υ}, the natural extension of f to υX [3, page 118]. Also, if Z_i $(i = 1, 2, \cdots)$ are zero sets, then

$$\mathrm{cl}_{\upsilon X} \bigcap_i Z_i = \bigcap_i \mathrm{cl}_{\upsilon X} Z_i .$$

Thus, the collections of zero sets of X having the c.i.p. are paired by extension with the collections of zero sets of υX having the c.i.p. Since every z-ultrafilter in υX with the c.i.p. has nonempty intersection, our lemma can be restated as follows: υX is a Lindelöf space if and only if every z-filter in υX with the c.i.p. has nonempty intersection. We have thus reduced the lemma to Problem 8H.5 of [3].

LEMMA 2. *If* X *is an* I*-space, then* υX *is a Lindelöf space.*

Proof. Let $\mathscr{F}$ be a z-filter with the c.i.p. Let $\mathscr{C}$ denote a maximal collection of closed sets with the c.i.p. containing $\mathscr{F}$. Let $\mathscr{C}'$ denote the collection of zero sets in $\mathscr{C}$. Using the maximality of $\mathscr{C}$ and an argument of the type appearing on page 30 of [3], we see that $\mathscr{C}'$ is a prime z-filter. Thus $Z(O^p) \subseteq \mathscr{C}' \subseteq Z(M^p)$ for some $p \in \beta X$, and the z-ultrafilter containing $\mathscr{C}'$ has the c.i.p. by Problem 7H.3 of [3]. By Lemma 1, υX is a Lindelöf space.

Our first theorem generalizes Theorem 2 in [1].

THEOREM 1. *A space* X *is both realcompact and an* I*-space if and only if* X *is a Lindelöf space.*

Proof. If X is a Lindelöf space, then X is realcompact, and as we remarked above, X is an I-space. The converse follows from Lemma 2. (J. E. Keesling has obtained an independent proof of Theorem 1.)

Received February 16, 1970.

Michigan Math. J. 17 (1970).

One might conjecture that X is an I-space if and only if υX is a Lindelöf space. This is not true. Exercise 5I on page 79 of [3] furnishes a counterexample. The space Ψ defined in this exercise is a completely regular, pseudocompact space that is not countably compact. The pseudocompactness of Ψ implies that $\upsilon\Psi$ is compact, hence a Lindelöf space. However, Ψ contains a closed, discrete, uncountable subset. An uncountable discrete space is not an I-space (provided the cardinal of the space is nonmeasurable [1]), and hence it follows that Ψ is not an I-space. However, we shall prove the following result.

THEOREM 2. *If* X *is normal and countably paracompact, then* X *is an* I-*space if and only if* υX *is a Lindelöf space.*

Proof. Let υX be a Lindelöf space, and let $\mathscr{C}$ denote a collection of closed sets with the c.i.p. Consider the collection of zero sets $\mathscr{F} = \left\{ Z \colon Z \supseteq \bigcap_i C_i,\ C_i \in \mathscr{C} \right\}$. The set $\mathscr{F}$ is a z-filter with the c.i.p. By Lemma 1, there exists a z-ultrafilter $\mathscr{U}$ with the c.i.p. containing $\mathscr{F}$. Let $\mathscr{U}'$ be some maximal collection of closed sets with the finite intersection property containing $\mathscr{U}$. Using Urysohn's Lemma, we see that $\mathscr{U}' \supseteq \mathscr{C}$. C. H. Dowker [2] characterizes the normal, countably paracompact spaces as follows: For every decreasing sequence of closed sets $F_1 \supseteq F_2 \supseteq \cdots$ with empty intersection, we can choose a sequence of neighborhoods $U_i \supseteq F_i$ such that $\bigcap_i U_i = \emptyset$. Using this characterization together with Urysohn's Lemma, we can see that $\mathscr{U}'$ has the c.i.p. The proof is complete.

An example of a normal space X that is not an I-space, while υX is a Lindelöf space, would be of great interest. By Theorem 2, such a space would fail to be countably paracompact, and it would give a negative answer to the famous question of Dowker [2]. That is, X would be a normal space such that $X \times I$ is not normal, where I is the unit interval.

REFERENCES

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