A NOTE ON FIBRATIONS AND CATEGORY

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Let cat X denote the Lusternik-Schnirelmann category of X as redefined by G. W. Whitehead [4], and suppose cat X is renormalized to take the value 0 on contractible spaces. Let p: $E \to B$ be a Hurewicz fibration, where B is arc-connected. Using an alternative definition of cat X that is equivalent for a large class of spaces, K. Varadarajan [3] proved the inequality

$$\text{cat } E \leq \text{cat } F + \text{cat } B + \text{cat } F \text{ cat } B, \tag{1}$$

where F denotes the fibre above some point * of B. Suppose that (B, *) is a closed cofibred pair. The purpose of this note is to prove the inequality

$$\text{cat } E \leq \text{cat } i + \text{cat } p + \text{cat } i \text{ cat } p, \tag{2}$$

where i: $F \to E$ denotes the injection and where the right-hand side is to be interpreted in the sense of the extension to maps of the (renormalized) definition of category due to Whitehead. (See [1].) Each map f: $Y \to B$ such that cat f < cat B, converted into a fibration, yields an example for which (2) is sharper than (1).

Let $\Pi^n X$ be the n-fold product of the based space X with itself, let $\Delta_X = \Delta_X^n$: $X \to \Pi^n X$ be the diagonal map, and let $j = j_X$: $T^n X \to \Pi^n X$ be the map that injects the fat wedge. Suppose that cat p = n - 1. We recall that under these conditions there exists a map ϕ: $E \to T^n B$ such that

$$j_B \cdot \phi \sim \Delta_B \cdot p. \tag{3}$$

Since $\Pi^n(p)$: $\Pi^n E \to \Pi^n B$ is a fibration and since $\Pi^n(p) \cdot \Delta_E = \Delta_B \cdot p$, the homotopy (3) may be lifted to a homotopy $\Delta_E \sim \phi'$: $E \to \Pi^n E$, where

$$\Pi^n(p) \cdot \phi' = j_B \cdot \phi. \tag{4}$$

Now suppose that cat i = m - 1, and choose θ: $F \to T^m E$ such that

$$j_E \cdot \theta \sim \Delta_E^m \cdot i. \tag{5}$$

Since the map $* \to B$ is a closed cofibration, it follows from [2; Theorem 12] that i is a cofibration. Hence the homotopy (5) can be extended to a homotopy $\Delta_E^m \sim \tau$: $E \to \Pi^m E$, where τ is such that

$$\tau \cdot i = j_E \cdot \theta. \tag{6}$$

Now Π^n is a functor that respects homotopies; hence we have the relations

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$$\Pi^n(\tau)\cdot\phi' \sim \Pi^n(\Delta_E^m)\cdot\phi' \sim \Pi^n(\Delta_E^m)\cdot\Delta_E^n = \Delta_E^{mn}.$$

Moreover, if $x \in E$, then, in view of (4), at least one coordinate of $\phi' x$ belongs to F. Hence, in view of (6), at least one coordinate of $\Pi^n(\tau)\cdot\phi' x$ is the base-point of E. Therefore $\Pi^n(\tau)\cdot\phi'$ can be factored through $T^{mn}E$. It follows that

$$\text{cat } E \leq mn - 1 = (m-1) + (n-1) + (m-1)(n-1),$$

as required.

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