

ON THE KRULL-SCHMIDT THEOREM FOR INTEGRAL GROUP REPRESENTATIONS OF RANK 1

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0. INTRODUCTION

Let R be a Dedekind domain with quotient field K ($\text{char } K = 0$). Let G be a finite group. An RG -module M is a finitely generated, torsionfree left R -module on which G acts from the left by R -homomorphisms

$$g: M \rightarrow M, \quad m \mapsto gm \quad (g \in G, m \in M).$$

We assume M to be imbedded in $KM = K \otimes_R M$, and we define $\text{rk } M = \text{Dim}_K KM$.

In [5], I. Reiner constructs counterexamples to the Krull-Schmidt Theorem for R - G -modules for the case where R is a ring of algebraic numbers in some algebraic number field K (integral at all places p dividing $|G|$), where the order of G is not a power of a prime, and where G contains a normal subgroup whose index is a prime. He points out that this method does not work for simple groups.

In this note, we provide easy counterexamples for groups G and rings R for which the ideal $|G|R$ is not primary in R ; in other words, $|G|R$ is contained in at least two different prime ideals. Thus we furnish counterexamples in the setting of [5], if G is not a p -group or if G is a p -group, but p decomposes in K .

We consider RG -modules of rank 1 and find conditions for an arbitrary RG -module M to be of the form $M' + M''$ with $\text{rk } M' = 1$. Our starting point is the following observation:

For each $U \leq G$, let $\mathbb{Z}[G/U]$ denote the $\mathbb{Z}G$ -module that is spanned as \mathbb{Z} -module by the left cosets $gU \in G/U$ with the obvious G -action. Then the trivial $\mathbb{Z}G$ -module $\mathbb{Z} = \mathbb{Z}[G/G]$ is a direct summand in $\mathbb{Z}[G/U]$ if and only if $U = G$ (in fact, every module $\mathbb{Z}[G/U]$ is indecomposable). But \mathbb{Z} is a direct summand in

$$\bigoplus_{p \mid |G|} \mathbb{Z}[G/G_p]$$

(where G_p is a p -Sylow subgroup in G), because the map $T_U: \mathbb{Z}[G/U] \rightarrow \mathbb{Z}$ ($gU \mapsto 1$) has no right inverse if $U \neq G$, whereas the map

$$\bigoplus_{p \mid |G|} T_{G_p}: \bigoplus_{p \mid |G|} \mathbb{Z}[G/G_p] \rightarrow \mathbb{Z}$$

has a right inverse.

Our results obviously do not apply in the situation where R is a discrete valuation ring, and they even seem to support the conjecture that the Krull-Schmidt

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Theorem holds in general for RG-modules in this case. But this conjecture is false, as was shown in [1] and [4].

I conjecture that the Krull-Schmidt Theorem holds for RG-modules if and only if each irreducible KG-module V for which $K^* \otimes V$ is not irreducible contains only projective RG-submodules, where K^* is the \mathfrak{p} -adic completion of K (\mathfrak{p} is the maximal ideal in R). One can easily see the sufficiency of this condition by extending the methods in [2]; but, up to now, I was unable to generalize the methods used in [1] and [4] to prove the necessity.

1. NOTATION

For each homomorphism $\phi: G \rightarrow R^\times$ from G into the group of units in R and each RG-module M , we define

$$M_\phi = \{m \in M \mid gm = \phi(g)m \text{ for each } g \in G\}.$$

Then M_ϕ is an RG-submodule of M , the inclusion $N \subseteq M$ implies that $N_\phi \subseteq M_\phi$, and for $M = M' \oplus M''$, we have that $M_\phi = M'_\phi \oplus M''_\phi$.

Each R -submodule of M_ϕ is also an RG-submodule. Therefore a well-known result [2, Theorem (22.5), p. 147] implies that M_ϕ is a direct sum of RG-modules of rank 1. Moreover, for each RG-submodule M' of M ($\text{rk } M' = 1$) there exists exactly one homomorphism $\phi: G \rightarrow R^\times$ such that $M' \subseteq M_\phi$.

There exists a natural RG-homomorphism

$$T_\phi: M \rightarrow KM_\phi, \quad m \mapsto \frac{1}{|G|} \sum_{g \in G} \phi(g)^{-1} gm$$

that is the identity on M_ϕ and maps M into $\frac{1}{|G|} M_\phi \subseteq KM_\phi$. Thus

$$M_\phi \subseteq T_\phi(M) \subseteq \frac{1}{|G|} M_\phi.$$

We define $s_\phi(M) = \max \{\text{rk } M' \mid M = M' \oplus M'', M' \subseteq M_\phi\}$; obviously, $s_\phi(M) \leq \text{rk } M_\phi$.

Finally, for each finitely-generated torsion R -module N , let $e(N)$ denote the minimal number of generators of N (as an R -module). The invariant-factor theorem [2, Theorem (22.12), p. 150] implies that

$$e(N) = \max \left\{ \text{Dim}_{R/\mathfrak{p}} R/\mathfrak{p} \otimes_R N \mid \mathfrak{p} \text{ is a maximal ideal in } R \right\};$$

in particular, $e(N' \oplus N'') = e(N') + e(N'')$ if and only if there exists a maximal ideal \mathfrak{p} with $e(N') = \text{Dim}_{R/\mathfrak{p}} R/\mathfrak{p} \otimes_R N'$ and $e(N'') = \text{Dim}_{R/\mathfrak{p}} R/\mathfrak{p} \otimes_R N''$.

2. RESULTS

THEOREM 1. *For each RG-module M, we have the relation*

$$s_\phi(M) + e(T_\phi(M)/M_\phi) = \text{rk } M_\phi.$$

Proof. By the invariant-factor theorem, we can find a basis m_1, \dots, m_n ($n = \text{rk } M_\phi$) of KM_ϕ , fractional ideals $\alpha_1, \dots, \alpha_n$, and ideals

$$\epsilon_1 \supseteq \epsilon_2 \supseteq \dots \supseteq \epsilon_n \supseteq |G| \cdot R$$

with $T_\phi(M) = \bigoplus \alpha_i m_i$ and $M_\phi = \bigoplus \alpha_i \epsilon_i m_i$ (these conditions determine the ideals ϵ_i uniquely). Assume that $\epsilon_j = R$ and $\epsilon_{j+1} \neq R$ for some $j \geq 0$, and choose a maximal \mathfrak{p} with $\epsilon_{j+1} \subseteq \mathfrak{p}$. Because

$$\bigoplus_1^j \alpha_i m_i \subseteq M_\phi \subseteq M,$$

the composition T_1 of T_ϕ with the projection $T_\phi(M) \rightarrow \bigoplus_1^j \alpha_i m_i$ is an idempotent endomorphism of M ; thus

$$M \cong \bigoplus_1^j \alpha_i m_i \oplus \text{Ke}T_1 \quad \text{and} \quad s_\phi(M) \geq j.$$

Moreover, with $M(\phi) = T_\phi(M)/M_\phi$, we have the relation $M(\phi) \cong \bigoplus_{j+1}^n R/\epsilon_i$; thus $n - j \geq e(M(\phi)) \geq \text{Dim}_{R/\mathfrak{p}} (\bigoplus_{j+1}^n R/\mathfrak{p}) = n - j$, that is, $e(M(\phi)) = n - j$ and $s_\phi(M) + e(M(\phi)) \geq n = \text{rk } M_\phi \geq e(M(\phi))$.

On the other hand, if $M = M' \oplus M''$ and $M' \subseteq M_\phi$ ($\text{rk } M' = s_\phi(M)$), then $\text{rk } M_\phi = s_\phi(M) + \text{rk } M''_\phi \geq s_\phi(M) + e(M''(\phi))$. But $M(\phi) = M'(\phi) \oplus M''(\phi)$ and $M'(\phi) = 0$; hence $e(M''(\phi)) = e(M(\phi))$ and $\text{rk } M_\phi = s_\phi(M) + e(M(\phi))$.

We remark that each maximal ideal \mathfrak{p} for which the equation

$$e(M(\phi)) = \text{Dim}_{R/\mathfrak{p}} (R/\mathfrak{p} \otimes M(\phi))$$

holds must contain $|G| \cdot R$, because $|G|$ annihilates $M(\phi)$ (unless we have the trivial situation $M(\phi) = 0$, in which case M_ϕ is a direct summand of M).

We mention a few applications.

COROLLARY 1. *For two RG-modules M' and M'', the following statements are equivalent:*

- (i) $s_\phi(M' \oplus M'') = s_\phi(M') + s_\phi(M'')$,
- (ii) $e(M'(\phi) \oplus M''(\phi)) = e(M'(\phi)) + e(M''(\phi))$,

(iii) *there exists a maximal ideal \mathfrak{p} (containing $|G| \cdot R$ in the case where $|G| \cdot R \neq R$) for which the relations*

$$e(M'(\phi)) = \text{Dim}_{R/\mathfrak{p}} (R/\mathfrak{p} \otimes M'(\phi)) \quad \text{and} \quad e(M''(\phi)) = \text{Dim}_{R/\mathfrak{p}} (R/\mathfrak{p} \otimes M''(\phi))$$

hold.

COROLLARY 2. *If $k[M]$ denotes the direct sum of k copies of M , then $s_\phi(k[M]) = k s_\phi(M)$.*

COROLLARY 3. *If $M = M' \oplus M''$, then M_ϕ is a direct summand of M if and only if M'_ϕ and M''_ϕ are direct summands of M' and M'' , respectively.*

For the next two applications, we need the following lemma.

LEMMA. *For each (maximal) ideal \mathfrak{p} satisfying the conditions $|G| R \subseteq \mathfrak{p} \subseteq R$, there exists an RG -module $M^\mathfrak{p}$ for which $M^\mathfrak{p}(\phi) \cong R/\mathfrak{p}$.*

Proof. Consider the ring RG ($RG \subseteq KG$) as a left RG -module. We have the relation $T_\phi(RG) = R \circ m$, where

$$m = \frac{1}{|G|} \sum \phi(g)^{-1} g \in KG,$$

and for $M^\mathfrak{p} = \mathfrak{p}m + RG \subseteq KG$, we have that $M^\mathfrak{p}(\phi) \cong R/\mathfrak{p}$.

Obviously, $e(M^\mathfrak{p}(\phi)) = \text{Dim}_{R/\mathfrak{q}}(R/\mathfrak{q} \otimes M^\mathfrak{q}(\phi))$ if and only if $\mathfrak{p} = \mathfrak{q}$.

COROLLARY 4. *Suppose M is an RG -module. The relation*

$$s_\phi(M \oplus M') = s_\phi(M) + s_\phi(M')$$

holds for all RG -modules M' if and only if $e(M(\phi)) = \text{Dim}_{R/\mathfrak{p}}(R/\mathfrak{p} \otimes M(\phi))$, for all maximal ideals containing $|G| R$.

THEOREM 2. *The equation $s_\phi(M' \oplus M'') = s_\phi(M') + s_\phi(M'')$ holds for all RG -modules M' and M'' if and only if $|G| R$ is primary in R , that is, if $|G| R$ is contained in at most one maximal ideal.*

3. REMARKS

a) If M is an indecomposable RG -module and M' is an arbitrary RG -module, one can define $s_M(M') = \max \{k \mid M' \cong k[M] \oplus M''\}$, and one can say that the Krull-Schmidt Theorem holds with respect to M if the relation

$$s_M(M' \oplus M'') = s_M(M') + s_M(M'')$$

holds for all RG -modules M' and M'' . Then Theorem 2 shows that the Krull-Schmidt Theorem is valid for RG -modules of rank 1 if R is a valuation ring, or if $R = \mathbb{Z}$ and G is a p -group, or, more generally, if R is some principal-ideal domain in $\mathbb{Q}(\zeta)$ (where ζ is a primitive p^n th root of unity) and G is a p -group. But, as pointed out in the introduction, it seems more interesting that the arguments leading to Theorem 2 provide easy counterexamples to the Krull-Schmidt Theorem, whenever there exist at least two different prime ideals \mathfrak{p} and \mathfrak{q} containing $|G| R$: for each homomorphism $\phi: G \rightarrow R^\times$, we have the relation $M^\mathfrak{p} \oplus M^\mathfrak{q} \cong R \oplus M'$, where G acts on R via ϕ ($g \cdot r = \phi(g)r$); but neither $M^\mathfrak{p}$ nor $M^\mathfrak{q}$ contains a direct summand of rank 1.

b) For each irreducible KG -module V and each RG -module M , one can consider the uniquely determined maximal submodule M_V for which KM_V is a direct sum of submodules isomorphic to V . Define

$$s_V(M) = \max \{ \text{rk } M' \mid M = M' \oplus M'', M' \subseteq M_V \}.$$

There exists a natural map $T_V: M \rightarrow KM_V$, $m \mapsto e_V m$, where e_V is the central idempotent in KG associated with V . Consider $M(V) = T_V(M)/M_V$. (In the case where $V = K$, we find that $g \cdot v = \phi(g)v$ ($v \in V$, $g \in G$); we considered these conditions in Section 2.)

The following statements still hold under these more general hypotheses.

$$(1) M(V) = M'(V) \oplus M''(V) \text{ if } M = M' \oplus M'';$$

(2) $M(V) = 0$ if and only if M_V is a direct summand of M (thus Corollary 3 still holds);

$$(3) s_V(M) + e(M(V)) \leq \text{rk } M_V.$$

Moreover, one can prove the following result.

THEOREM 3. *Consider the following three statements.*

(i) *If M, M' are two RG -submodules of V , then $M = \alpha M'$ for some ideal $\alpha \subseteq R$.*

(ii) *If M ($M \neq 0$) is an RG -submodule of V and \mathfrak{p} is a maximal ideal in R , then $R/\mathfrak{p} \otimes M$ is an irreducible $(R/\mathfrak{p})G$ -module.*

(iii) *For each RG -module M , we have the relation $s_V(M) + e(M(V)) = \text{rk } M_V$.*

The implications (i) \Leftrightarrow (ii) \Rightarrow (iii) hold in general, and the implication (ii) \Leftarrow (iii) holds if K is a splitting field for G .

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