THE HAUSDORFF DIMENSION OF CERTAIN SETS OF NONNORMAL NUMBERS

C. M. Colebrook

1. THE GENERATION OF MEASURES

Let $C[0, 1)$ denote the collection of continuous, periodic functions (with period 1) on the reals, and let s be a fixed integer $(s \geq 2)$. We say that a real number x is *ν-normal* (to the base s) if the sequence $\{s^n x\}$ has the distribution ν in the sense that

$$(1.1) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(s^k x) = \int_0^1 f \, d\nu = \nu(f) \qquad \text{for each } f \in C[0, 1).$$

Here, ν denotes a probability measure on $[0, 1)$ that is invariant under the transformation $T: y \to Ty = sy - [sy]$. In other words, ν is the weak* limit of the measures μ_n defined by the relations

$$(1.2) \qquad \mu_n(f) \equiv \mu_{n,x}(f) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x).$$

We say that x *generates* ν (to the base s). The space of all these T-invariant probability measures, together with the weak* topology, is denoted by $I(s)$. Observe that for each positive integer n we have the inclusion $I(s) \subset I(s^n)$. One measure in $I(s)$ is the ordinary Lebesgue measure λ, and x is normal to the base s in the classical sense precisely when x is λ-normal to the base s.

Consider a measure ν in $I(s)$, and let x generate ν (to the base s). As is well known (see for instance [5]), relation (1.1) also holds when f is bounded and the set of discontinuities of f has ν-measure 0. In particular, the relation (1.1) holds when f is the characteristic function of an interval $[\alpha, \beta)$. That is,

$$(1.3) \qquad \begin{aligned} &\lim_{n \to \infty} \frac{1}{n} (\text{number of } k \ (0 \leq k \leq n-1), \text{ for which } T^k x \in [\alpha, \beta)) \\ &\qquad \equiv \lim_{n \to \infty} \mu_n([\alpha, \beta)) = \nu([\alpha, \beta)), \end{aligned}$$

provided $\nu(\{\alpha\}) = 0 = \nu(\{\beta\})$. Also, it is known that if the relation (1.3) holds for some point x and for all choices of α and β such that $\nu(\{\alpha\}) = 0 = \nu(\{\beta\})$, then x is ν-normal (to the base s) (see [5]). In fact, for x to be ν-normal, it suffices to require that condition (1.3) hold only for intervals of the form $[as^{-n}, (a+1)s^{-n})$,

Received February 15, 1969.

This work was supported in part by a Faculty Research Fellowship from the State University of New York.

Michigan Math. J. 17 (1970).

where a and n are positive integers ($1 \leq a \leq s^n - 2$). In this connection, we mention that for each T-invariant measure ν and each positive integer n, the relations

$$\nu(\{as^{-n}\}) = 0 \qquad (a = 1, 2, \cdots, s^n - 1)$$

hold. To see this, we observe that the sets $T^{-i}\{as^{-n}\}$ ($i = 0, 1, \cdots$) are disjoint; therefore

$$\nu(\{as^{-n}\}) = \frac{1}{m}\,\nu\left(\bigcup_{i=0}^{m-1} T^{-i}\{as^{-n}\}\right) \leq \frac{1}{m}$$

for $m = 1, 2, \cdots$. (However, $\nu(\{0\})$ may be positive or 0.) We conclude that ν-normality of x to the base s is equivalent to the validity of condition (1.3) with $[\alpha, \beta) = [as^{-n}, (a+1)s^{-n})$, where $a = 1, 2, \cdots, s^n - 2$ ($n = 1, 2, \cdots$). If $\nu(\{0\}) = 0$, then we may also include 0 and $s^n - 1$ among the values of a.

Let c be a positive integer, and let f_a denote the characteristic function of the interval $[as^{-c}, (a+1)s^{-c})$ ($a = 0, 1, \cdots, s^c - 1$). We shall write W(c) for the collection of the s^c characteristic functions f_a ($a = 0, 1, \cdots, s^c - 1$), and we shall denote $\bigcup_{c=1}^{\infty} W(c)$ by W. From the discussion above we see that we can obtain the distribution properties of the number x by considering the behavior of the sequence $\{\mu_{n,x}(f)\}$ for $f \in W$ instead of $f \in C[0, 1)$.

2. SOME TYPES OF NONNORMALITY: REGULARITY AND SIMPLE REGULARITY

Let ν be a measure in I(s). The number x is said to be *simply ν-regular* (to the base s) if condition (1.3) holds for the intervals

$$[\alpha, \beta) = [as^{-1}, (a+1)s^{-1}) \qquad (a = 0, 1, \cdots, s - 1),$$

or equivalently, if condition (1.1) holds for $f \in W(1)$. We denote the set of all such x by $F(\nu, s)$. The number x is said to be *ν-regular* (to the base s) if

$$x \in \bigcap_{c=1}^{\infty} F(\nu, s^c) = G(\nu, s).$$

Obviously, one can also define simple ν-regularity and ν-regularity in terms of the digits in the expansion of x to the base s (see [1] for example). We observe that a number that is ν-regular to the base s is also ν-normal to the base s^n ($n = 1, 2, \cdots$). The converse holds if $\nu(\{0\}) = 0$.

3. HAUSDORFF DIMENSION

Throughout this paper, dim A denotes the usual Hausdorff dimension of $A \subset [0, 1)$. Let ν be a measure in I(s). H. G. Eggleston [2] proved the relation

$$\dim F(\nu, s) = -\sum_{f \in W(1)} \nu(f) \log_s \nu(f).$$

(We shall always set $p \log_s p = 0$ if $p = 0$.) Let $h(\nu, c)$ be defined by

$$h(\nu, c) = - \sum_{f \in W(c)} \nu(f) \log_{s^c} \nu(f) . \tag{3.1}$$

Then

$$\dim G(\nu, s) \leq \inf_c \dim F(\nu, s^c) = \inf_c h(\nu, c).$$

From well-known work on entropy (see for instance [4, page 48]), we conclude that

$$\inf_c h(\nu, c!) = \lim_{c \to \infty} h(\nu, c).$$

We denote this limit by $h(\nu)$. Thus

$$\dim G(\nu, s) \leq h(\nu).$$

In Section 7, we show that in fact equality holds.

Let us now choose a number x. In general, the weak* limit of $\{\mu_{n,x}\}$ does not exist, so that x fails to have a distribution to the base s. Let $V'(x, s)$ denote the set of probability measures that are weak* accumulation points of the sequence $\{\mu_{n,x}\}$. We shall say x *generates* $V'(x, s)$ (to the base s). Clearly $V'(x, s) \subset I(s)$. It is easy to show that $V'(x, s)$ is always nonempty, closed, and connected in $I(s)$.

Let V be any nonempty, closed, connected subset of $I(s)$. We shall be interested in the set $G(V, s)$, defined as the set of all points x that generate V to each base $s, s^2, \cdots$. In other words, $x \in G(V, s)$ if and only if

$$V'(x, s^c) = V \qquad (c = 1, 2, \cdots) .$$

In the case where $V = \{\nu\}$ and $\nu(\{0\}) = 0$, one easily sees that this definition of $G(V, s)$ agrees with the definition of $G(\nu, s)$ given previously. (If $\nu(\{0\}) \neq 0$ then $G(\{\nu\}, s) \supset G(\nu, s)$.) In particular, consider the Lebesgue measure λ. It is known (see for instance [1]) that every number that is normal to the base s in the classical sense is also normal to the base s^c $(c = 2, 3, \cdots)$. Hence $G(\lambda, s)$ is the set of all numbers normal to the base s. This set is known to have Lebesgue measure 1 (see [1]). Thus the Lebesgue measure of $G(V, s)$ is 0 unless $V = \{\lambda\}$.

The following questions now arise. If V is some nonempty, closed, connected subset of $I(s)$, is $G(V, s)$ nonempty? Further, can we obtain an estimate of the size of $G(V, s)$?

In Section 6, we construct a nonempty subset of $G(V, s)$. Further, in Section 7, we show that the Hausdorff dimension of $G(V, s)$ is given by $\inf_{\nu \in V} h(\nu)$.

4. AN UPPER BOUND ON THE HAUSDORFF DIMENSION OF G(V, s)

Let c be a positive integer, and suppose $x \in [0, 1)$. We define $\mu_n^{(c)}$ by the condition

$$\mu_n^{(c)}(f) \equiv \mu_{n,x}^{(c)}(f) = \frac{1}{n} \sum_{i=0}^{n-1} f(T^{ic} x) \qquad (f \in W).$$

For each set $V \subset I(s^c)$, we define $F(V, s^c)$ as the set of all points $x \in [0, 1)$ for which the set

$$\{(\nu(f), f \in W(c)), \nu \in V\},$$

regarded as a subset of $[0, 1]^{s^c}$, is the set of accumulation points of

$$\{(\mu_{n,x}^{(c)}(f), f \in W(c))\}.$$

We note that the relations

$$\mu_{cn,x} \equiv \mu_{cn} = \frac{1}{c}(\mu_n^{(c)} + T\mu_n^{(c)} + \cdots + T^{c-1}\mu_n^{(c)})$$

hold, where $T\mu$ is defined by $T\mu(f) = \mu(Tf)$ and Tf is given by $Tf(y) = f(Ty)$ ($y \in [0, 1)$). Also, we see that

$$\lim_{j\to\infty} T^i \mu_{n_j}^{(c)} = T^i \lim_{j\to\infty} \mu_{n_j}^{(c)}$$

whether either limit exists. From these equations, we obtain the relation

$$\lim_{j\to\infty} \mu_{cn_j} = \frac{1}{c}\sum_{i=0}^{c-1} T^i \lim_{j\to\infty} \mu_{n_j}^{(c)}. \tag{4.1}$$

Equation (4.1) implies the inclusion $V'(x, s) \supset V'(x, s^c) \cap I(s)$. Further, an application of (4.1) to the case where $x \in F(V, s^c)$ and $V \subset I(s)$ shows that $x \in F(V, s)$. In other words, $F(V, s) \supset F(V, s^c)$ if $V \subset I(s)$. We can now prove the following lemma.

(4.2) LEMMA. *For each* $V \subset I(s)$, *we have the inclusion*

$$G(V, s) \supset \bigcap_{c=1}^{\infty} F(V, s^c).$$

Proof. Let x be a fixed point of $\bigcap_{j=1}^{\infty} F(V, s^j)$. Let c be a positive integer. Since $x \in F(V, s^c)$, it follows that $V \supset V'(x, s^c)$. Let ν be a measure in V. For each m ($m = 1, 2, \cdots$), there is a sequence $\{n_{j,m}\}$ such that

$$\lim_{j\to\infty} \mu_{n_{j,m}}^{(mc)}(f) = \nu(f) \qquad (f \in W(mc)).$$

Such a sequence exists, because $x \in F(V, s^{mc})$. Since $\nu \in I(s)$, we can use relation (4.1) to obtain the relation

$$\lim_{j\to\infty} \mu_{mn_{j,m}}^{(c)}(f) = \nu(f) \qquad (f \in W(mc)).$$

We note also that each function in $W(p)$ is the sum of functions in $W(q)$, whenever $q > p$. Thus we can use a diagonal procedure on $\{\{mn_{j,m}\}, m = 1, 2, \cdots\}$ to find a sequence $\{n_j\}$ with the property that

$$\lim_{j\to\infty} \mu_{n_j}^{(c)}(f) = \nu(f) \qquad (f \in \bigcup_{m=1}^{\infty} W(mc)).$$

Hence $\nu \in V'(x, s^c)$, and we have the inclusion $V'(x, s^c) \supset V$.

We have proved that $V'(x, s^c) = V$ for $c = 1, 2, \cdots$. But this is precisely the requirement for x to belong to $G(V, s)$.

On the other hand, suppose that $x \in G(V, s)$. Let c be a fixed positive integer. Then x also belongs to $F(V', s^c)$, where V' is some closed connected set such that $V \subset V' \subset I(s^c)$. Thus

$$G(V, s) \subset \bigcup F(V', s^c),$$

where the union is taken over all closed connected sets V' with the property that $V \subset V' \subset I(s^c)$. Now, Theorem 4 of Volkmann [6] gives the Hausdorff dimension of this union of sets as

$$\dim \bigcup F(V', s^c) = \inf_{\nu \in V} h(\nu, c).$$

It follows that $\dim G(V, s) \leq \inf_{\nu \in V} h(\nu, c)$ $(c = 1, 2, \cdots)$; therefore

$$\dim G(V, s) \leq \inf_{c} \inf_{\nu \in V} h(\nu, c) = \inf_{\nu \in V} \inf_{c} h(\nu, c) \leq \inf_{\nu \in V} h(\nu). \tag{4.3}$$

In the proof of Theorem (7.2) we shall show that

$$\dim \bigcap_{c=1}^{\infty} F(V, s^c) \geq \inf_{\nu \in V} h(\nu). \tag{4.4}$$

By (4.2) and (4.4), we have the inequality

$$\dim G(V, s) \geq \inf_{\nu \in V} h(\nu), \tag{4.5}$$

and from (4.3) and (4.5) we conclude that

$$\dim G(V, s) = \inf_{\nu \in V} h(\nu).$$

In particular, let us examine the case where $V = \{\nu\}$. If $\nu(\{0\}) = 0$, then ν-regularity to the base s, and ν-normality to all the bases s^n $(n = 1, 2, \cdots)$ are equivalent. For instance, this condition is satisfied if ν is ergodic with respect to T (except for the trivial case, where ν is the point measure ε_0 defined by $\varepsilon_0(f) = f(0)$). Let ν be ergodic $(\nu \neq \varepsilon_0)$. Then $G(\{\nu\}, s)$ is the set of all numbers that are ν-normal to each base $s, s^2, \cdots$. Therefore $h(\nu)$ is the Hausdorff dimension of the set of all numbers that are ν-normal to each base $s, s^2, \cdots$.

Let $s \geq 2$ be a fixed integer. Let V be a fixed, closed, connected, nonempty subset of $I(s)$. In the next section, we shall construct a subset R of $G(V, s)$. Then, in Section 7, we show that $\dim R \geq \inf_{\nu \in V} h(\nu)$.

5. THE DEFINITION OF $R(n, N, \nu)$

For each measure $\nu \in I(s)$, for each positive integer N, and for $f \in W(n)$ (n a positive integer), let

$$\Phi(f, N, \nu) = [\nu(f)N] + \xi(f).$$

Here $\xi(f) = 1$ for each $f \in W(n)$ with the exception of one function $\bar{f}$ chosen to satisfy the condition $\nu(\bar{f}) = \max_{f \in W(n)} \nu(f)$. Note that $\nu(\bar{f}) \geq s^{-n}$ and $\nu(\bar{f}) \geq N^{-1} s^n$ if $N \geq s^{2n}$. We choose $\xi(\bar{f})$ so that $N = \sum_{f \in W(n)} \Phi(f, N, \nu)$. It is clear that $\Phi(f, N, \nu)$ is positive provided $N \geq s^{2n}$. Also $\Phi(\bar{f}, N, \nu) \geq [\nu(\bar{f})N] - (s^n - 1)$, thus if $N \geq s^{2n}$ then $\Phi(\bar{f}, N, \nu) \geq 1$. From now on we shall assume that $N \geq s^{2n}$.

We consider the set $\{0, 1, \cdots, s^n - 1\} = \{a: f_a \in W(n)\}$, and the permutations with repetitions of the elements of this set that we obtain by repeating each a exactly $\Phi(f_a, N, \nu)$ times. That is, each permutation is an ordering of N integers y $(0 \leq y \leq s^n - 1)$. Clearly, there are

$$S(n, N, \nu) = \frac{N!}{\prod \Phi(f, N, \nu)!}$$

such permutations. (Here the product is taken over all $f \in W(n)$.) For each permutation $(y_1, y_2, \cdots, y_n)$ $(0 \leq y_i \leq s^n - 1)$, we consider the associated interval $[y, y + s^{-nN})$, where

$$s^{nN} y = s^{n(N-1)} y_1 + s^{n(N-2)} y_2 + \cdots + y_N.$$

Obviously, different permutations are associated with disjoint intervals. We denote the union of these disjoint, half-open intervals by $R(n, N, \nu)$. Each point in $R(n, N, \nu)$ "approximately" generates ν in the sense indicated in the following lemma.

MAIN LEMMA. *Let* $\nu \in I(s)$ *be a fixed measure, and let* c, n, *and* N *be positive integers* (c *divides* n; $N \geq s^{2n}$). *For each* $x \in R(n, N, \nu)$ *and each* $f \in W(c)$, *we have the inequality*

$$\left| \sum_{i=0}^{c^{-1}nN-1} f(T^{ic}x) - c^{-1} nN \nu(f) \right| < 2c^{-1} ns^n.$$

Notation. Let $f \in W(c)$ be fixed. For all nonnegative integers i and n (c divides n), we define $w(f, i, n)$ to be the set of all functions $g \in W(n)$ for which $g(x) = 1$ implies $f(T^{ic}x) = 1$. We have the relation

$$\sum_{g \in w(f,i,n)} g(x) = f(T^{ic}x) = T^{ic}f(x).$$

We note that the set $\{w(f, i, n): f \in W(c)\}$ is a partition of $W(n)$ with the property that

$$\sum_{g \in w(f,i,n)} \nu(g) = \nu(T^{ic}f) = \nu(f), \tag{5.1}$$

because ν is invariant under T. Note that w(f, i, n) is empty if $ic \geq n$; otherwise w(f, i, n) has s^{n-c} elements.

Proof of the lemma. Let $x \in R(n, N, \nu)$ and $f \in W(c)$ be fixed. From the definition of $R(n, N, \nu)$, it follows that the number of values of j $(0 \leq jc < nN)$, for which $f(T^{jc}x) = 1$, is given by the sum

$$\sum_{i=0}^{c^{-1}n-1} \sum_{g \in w(f,i,n)} \Phi(g, N, \nu).$$

Now relation (5.1) implies that

$$\sum_{g \in w(f,i,n)} \Phi(g, N, \nu) = \sum_{g \in w(f,i,n)} \nu(g)N + \theta(f, i, n) = N\nu(f) + \theta(f, i, n),$$

where

$$\theta(f, i, n) = \sum_{g \in w(f,i,n)} (\Phi(g, N, \nu) - \nu(g)N).$$

The definition of $\Phi(g, N, \nu)$ implies the inequalities

$$|\theta(f, i, n)| \leq \sum_{g \in w(f,i,n)} |[\nu(g)N] - \nu(g)N + \xi(g)| < 2s^n.$$

Thus

$$\left|\sum_{j=0}^{c^{-1}nN-1} f(T^{jc}x) - c^{-1}nN\nu(f)\right| = \left|\sum_{i=0}^{c^{-1}n-1}\left(\sum_{g \in w(f,i,n)} (\Phi(g, N, \nu) - N\nu(g))\right)\right|$$

$$= \left|\sum_{i=0}^{c^{-1}n-1} \theta(f, i, n)\right| < 2c^{-1}ns^n,$$

and the lemma is proved.

Let $\{n_p\}$, $\{N_p\}$, and $\{t_p\}$ be sequences of integers $(N_p \geq s^{2n_p})$, and put $R_p = T^{-t_{p-1}}R(n_p, N_p, \nu_p)$, where $\{\nu_p\}$ is a sequence of measures in I(s). For convenience, we write S_p for $S(n_p, N_p, \nu_p)$. We define R to be the intersection of all R_p:

$$R = \bigcap_{p=1}^{\infty} R_p.$$

The set R is always nonempty, provided $t_p = n_pN_p + t_{p-1}$ and $t_0 = 0$. If $x \in R$, then $T^{t_{p-1}}x \in R(n_p, N_p, \nu_p)$ $(p = 1, 2, \cdots)$. By the Main Lemma, $T^{t_{p-1}}x$ "approximately" generates ν_p. In Section 6, we shall choose a sequence $\{\nu_p\}$ that is dense in V. Then we shall choose sequences $\{n_p\}$ and $\{N_p\}$ such that each point in R generates just V to each base s, s^2, $\cdots$. Clearly, the set R is of the form considered by Eggleston [3, Theorem 4].

Let $h \geq 0$ be prescribed. Suppose we show that the series

$$\sum_{p=1}^{\infty} \frac{n_p N_p s^{\alpha t_p}}{\prod_{q=1}^{p} s_q} \tag{5.2}$$

converges for all $\alpha < h$. By Eggleston's theorem, it follows that the Hausdorff dimension of R is at least h. In Section 7, we shall show that the series (5.2) converges for all $\alpha < \inf_{\nu \in V} h(\nu)$. Henceforth we denote $\inf_{\nu \in V} h(\nu)$ by h.

Let $\{n_p\}$ and $\{N_p\}$ be two unbounded, nondecreasing sequences of positive integers ($N_p \geq s^{2n_p}$) that satisfy the additional conditions (here $t_0 = 0$, $t_p = n_p N_p + t_{p-1}$)

(i) $n_p N_p = o(t_{p-1})$,

(ii) $n_p = m_p!$ (m_p an integer),

(iii) n_{p+1} divides t_p.

(Such a pair of sequences can be constructed by induction on p, starting with arbitrary values $m_1! = n_1$ and $N_1 \geq s^{2n_1}$.)

The following readily established lemma will be used to show convergence properties of certain functions of $\{n_p\}$ and $\{N_p\}$.

LEMMA. *If* $\{u_p\}$ *and* $\{v_p\}$ *are sequences of positive terms such that* $\{v_p\}$ *diverges and* $\lim_{p \to \infty} v_p/u_p = 0$, *then*

$$\lim_{n \to \infty} \frac{v_1 + v_2 + \cdots + v_n}{u_1 + u_2 + \cdots + u_n} = 0.$$

6. A DENSE SEQUENCE OF MEASURES IN V

Let $\{a(n)\}$ and $\{b(n)\}$ ($a(n) > b(n)$) be fixed sequences decreasing to zero, and let $\{c(n)\}$ be an increasing sequence of integers. For each $\nu \in I(s)$ and each positive integer n, we define a neighborhood $U_n(\nu)$ with center ν by the formula

$$U_n(\nu) = \left\{ \mu \in I(s) \colon \sum_{f \in W(c(n))} |\mu(f) - \nu(f)| < b(n) \right\}.$$

For each positive integer n, the compact set V admits a finite cover by sets of the form $U_n(\nu)$ ($\nu \in V$). Allowing repetitions, we may assume that each such cover can be written as $\{U_n(\nu(j, n)), 1 \leq j \leq j_n\}$, where j_n is finite. Further, we may assume that

$$U_n(\nu(j, n)) \cap U_n(\nu(j-1, n)) \neq \emptyset \quad (j = 2, 3, \cdots, j_n) \quad \text{and}$$

$$U_n(\nu(j_n, n)) \cap U_{n+1}(\nu(1, n+1)) \neq \emptyset.$$

(Here we have used the connectivity of V.) For each n, we henceforth assume a fixed cover.

Let $T_0 = 0$, $T_M = T_{M-1} + j_M$. For each positive integer r, let $M \equiv M_r$ be determined by the inequalities $T_{M-1} < r \leq T_M$. We choose the integer $p_r \equiv p(r)$ so that $p(0) = 0$ and

$$(6.1) \qquad t_{p(r-1)}(1 - b(M)) < t_{p(r)}(a(M) - b(M)) - 2 \sum_{q=p(r-1)+1}^{p(r)} n_q s^{n_q}.$$

Such a choice of p(r) is possible, because $a(M) > b(M)$, $t_p \to \infty$, and $s^{n_q} = o(N_q)$. By the lemma, the last condition implies that $\sum_{q=1}^{p} n_q s^{n_q} = o(t_p)$.

For each $r = T_{M-1} + j$ $(1 \leq j \leq j_M)$, set $U_r = U_M(\nu(j, M))$. First we order the centers of the neighborhoods U_r, allowing repetitions. For each p $(p = 1, 2, \cdots)$, choose r so that $p_{r-1} < p \leq p_r$. We define ν_p to be the center of U_r. One can easily verify that the sequence $\{\nu_p\}$ is dense in V.

We now show that with our choice of the sequences $\{\nu_p\}$, $\{n_p\}$, and $\{N_p\}$, every point of R generates the prescribed set V to each base $s, s^2, \cdots$.

THEOREM.

$$R \subset \bigcap_{c=1}^{\infty} F(V, s^c).$$

Proof. We shall show that the following two propositions hold for each $x \in R$ and each positive integer c.

(A) *For each* $\nu \in V$, *there exists a sequence* $\{k_j\}$ *such that*

$$(6.2) \qquad \lim_{j \to \infty} \frac{1}{k_j} \sum_{i=0}^{k_j - 1} f(T^{ic} x) = \nu(f) \qquad (f \in W(c)).$$

(B) *If* $\{k_j\}$ *is a sequence for which* (6.2) *holds for some* $\nu' \in I(s^c)$, *then there exists* $\nu \in V$ *such that* $\nu(f) = \nu'(f)$, *for all* $f \in W(c)$.

In other words, in (A) we show that $x \in F(V', s^c)$ for some $V' \supset V$, and in (B) we show that in fact $x \in F(V, s^c)$.

Proof of (A). Let x, ν, and c be fixed ($x \in R$; $\nu \in V$; c a positive integer). By construction, $\{U_r: T_{M-1} < r \leq T_M\}$ is a cover of V for each M $(M = 1, 2, \cdots)$. Thus, for each M, there exists at least one integer r in the range $T_{M-1} < r \leq T_M$ such that $\nu \in U_r$. Let $r \equiv r(\nu, M)$ be an integer. Let M be so large $(M \geq M_0$, say) that $c(M) \geq c$ and that $n_{p(r)}$ is a multiple of c when $r \geq r(\nu, M_0)$ (here we use the relation $n_p = m_p!$). We show that the sequence

$$\{n = c^{-1} t_{p(r)}: r = r(\nu, M), \ M \geq M_0\}$$

satisfies condition (6.2). Note that c divides $t_{p(r-1)}$, because c divides $n_{p(r)}$. For $n = c^{-1} t_{p(r)}$ and $f \in W(c)$, we have the inequalities

$$c\left|\sum_{i=0}^{n-1} f(T^{ic}x) - n\nu(f)\right|$$

$$\leq \left|c\sum_{i=0}^{n-1} f(T^{ic}x) - t_{p(r-1)}\nu(f)\right| + (t_{p(r)} - t_{p(r-1)})|\nu_{p(r)}(f) - \nu(f)|$$

$$+\left|c\sum_{i=0}^{c^{-1}n_{p(r)}N_{p(r)}-1} f(T^{t_{p(r-1)}+ic}x) - n_{p(r)}N_{p(r)}\nu_{p(r)}(f)\right|$$

$$\leq t_{p(r-1)} + (t_{p(r)} - t_{p(r-1)})b(M) + 2\sum_{p=p(r-1)+1}^{p(r)} n_p s^{n_p}.$$

Here we used the Main Lemma and the fact that $c \leq c(M)$. From (6.1), we obtain the relation

$$c\left|\sum_{i=0}^{n-1} f(T^{ic}x) - n\nu(f)\right| < t_{p(r)}\, a(M).$$

But $nc = t_{p(r)}$ and $a(M) \to 0$ as n (and thus M) increases. Hence (A) is proved.

Proof of (B). Let x ($x \in R$) and the integer c be fixed. Suppose that $\{k_j\}$ is a sequence for which the limit on the left-hand side of the equation (6.2) exists (and equals d_f, say) for all $f \in W(c)$. For each positive integer j, there exist unique integers $r = r(j)$ and $q = q(j)$ such that

$$t_{p(r)} < t_{q-1} \leq ck_j < t_q \leq t_{p(r+1)}.$$

Define $M = M(j)$ by the condition $T_{M-1} < r(j) \leq T_M$. From now on, let j be large enough so that $c(M(j)) \geq c$ and so that c divides $n_{p(r)}$ if $r \geq r(j)$ (then c divides $t_{p(r)}$). For $r = r(j)$, we have the inequality

$$k_j c\,|\nu_{p(r)}(f) - d_f| \leq c\left|\sum_{i=0}^{k_j-1} f(T^{ic}x) - k_j d_f\right| + \left|c\sum_{i=0}^{c^{-1}t_{p(r)}-1} f(T^{ic}x) - t_{p(r)}\nu_{p(r)}(f)\right|$$

$$+\sum_{q=p(r)+1}^{q(j)-1}\left(n_q s^{n_q} + n_q N_q\,|\nu_{p(r)}(f) - \nu_{p(r+1)}(f)|\right) + n_{q(j)}N_{q(j)}.$$

Here we have used the Main Lemma with $\nu = \nu_{p(r+1)} = \nu_q$ for

$$p(r) + 1 \leq q \leq p(r+1).$$

The definition of R and the hypothesis on $\{k_j\}$ imply that the right-hand side of the expression above is $o(k_j)$. Thus

$$\lim_{r=r(j),\, j\to\infty} \nu_{p(r)}(f) = d_f \qquad (f \in W(c)).$$

But $\{\nu_p\}$ is dense in V. It follows that the sequence $\{\nu_{p(r(j))}(f),\ f \in W(c)\}$, which, as we have just shown, converges in $[0, 1]^{s^c}$, has as limit the point $(\nu(f),\ f \in W(c))$, where ν is some measure in V.

From Lemma (4.2) and the theorem, we conclude that $R \subset G(V, s)$.

7. A LOWER BOUND ON THE HAUSDORFF DIMENSION OF G(V, s)

We find a lower bound for the dimension of R by the method outlined in Section 5.

(7.1) LEMMA. *Let* $a_1, a_2, \cdots, a_m$ *and* $b_1, b_2, \cdots, b_m$ *be nonnegative numbers such that* $a_1 + a_2 + \cdots + a_m = b_1 + b_2 + \cdots + b_m = k$, *where* $k \geq 1$. *If* $0 < a_i - b_i \leq 1$ $(i = 2, 3, \cdots, m)$ *and* $a_1 \geq 1$, *then*

$$\sum_{i=1}^{m} (a_i \log_e a_i - b_i \log_e b_i) \leq m \log_e k.$$

Proof. Let $\theta(x) = \sum_{i=1}^m (xa_i + (1-x)b_i)\log_e(xa_i + (1-x)b_i)$. By the Mean-Value Theorem, $\theta(1) - \theta(0) = \theta'(\xi)$ for some $\xi \in (0, 1)$. But

$$\theta'(\xi) = \sum_{i=1}^{m} (a_i - b_i)\log_e(\xi a_i + (1-\xi)b_i) \leq \sum \log_e(\xi a_i + (1-\xi)b_i),$$

where the summation extends over only those i for which $\xi a_i + (1-\xi)b_i \geq 1$. Certainly, the inequality $\xi a_i + (1-\xi)b_i \leq k$ holds for all i (recall that

$$1 \leq a_1 \leq xa_1 + (1-x)b_1 \leq b_1);$$

thus

$$\theta'(\xi) \leq m \log_e k.$$

(7.2) THEOREM.

$$\dim R \geq \inf_{\nu \in V} h(\nu).$$

If $\inf_{\nu\in V} h(\nu) = 0$, the result is trivial. Suppose $h = \inf_{\nu \in V} h(\nu) > 0$. As described in Section 5, we shall show that the series $\sum u_p$ converges for $0 < \alpha < h$, where

$$\log_s u_p = n_p N_p + \alpha t_p - \sum_{q=1}^{n} \log_s S_q.$$

First we need a bound for S_q.

(7.3) LEMMA. *Let* n *and* N *be integers* ($N \geq s^{2n}$), *and let* $\nu \in I(s)$. *Let* $S = S(n, N, \nu)$ *be as in Section 5. Then*

$$\log_s S - nNh(\nu, n) > -\frac{3}{2} N^{1/2} \log_s 2\pi N - \frac{1}{12}\frac{s^{2n}}{N} \log_s e .$$

Proof of lemma. Let $\Psi(x)$ be defined by the condition

$$\Gamma(x+1) = x! = x^x e^{-x+\Psi(x)} \qquad (x > 0).$$

By virtue of the well-known relation

$$\Psi''(x) = -\frac{1}{x} + \log_e (\Gamma(x+1))'' = -\frac{1}{x} + \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} \leq -\frac{1}{x} + \int_x^{\infty} \frac{1}{u^2} = 0$$

(see [7, p. 241]), $\Psi(x)$ is a convex function. Thus

$$(7.4) \qquad \sum_{f \in W(n)} \Psi(\Phi(f, N, \nu)) \leq s^n \Psi\left(s^{-n} \sum_{f \in W(n)} \Phi(f, N, \nu) \right) = s^n \Psi(s^{-n} N).$$

Clearly, the identities

$$\log_s S = \log_s N! - \sum_{f \in W(n)} \log_s (\Phi(f, N, \nu)!) = N \log_s N - (\log_s e)(N - \Psi(N))$$

$$- \sum_{f \in W(n)} (\Phi(f, N, \nu) \log_s \Phi(f, N, \nu) - (\log_s e)\Phi(f, N, \nu) + (\log_s e)\Psi(\Phi(f, N, \nu)))$$

hold.

Now we use Lemma (7.1) with $m = s^n$, $k = N$. We set $a_f = \Phi(f, N, \nu)$ ($\Phi(f, N, \nu) \geq 1$) and $b_f = \nu(f)N$ ($f \in W(n)$), and we observe that

$$\theta(0) = (N \log_s N - nNh(\nu, n)) \log_e s .$$

Thus

$$\theta(1) \log_s e = \sum_{f \in W(n)} \Phi(f, N, \nu) \log_s \Phi(f, N, \nu) < -nNh(\nu, n) + N \log_s N + s^n \log_s N .$$

From this inequality and inequality (7.4), we obtain the condition

$$\log_s S > nNh(\nu, n) - s^n \log_s N + (\Psi(N) - s^n \Psi(s^{-n} N)) \log_s e .$$

The Stirling bounds on $\Psi(x)$ are

$$\frac{1}{2} \log_e 2\pi x < \Psi(x) < \frac{1}{12x} + \frac{1}{2} \log_e 2\pi x$$

(see [7, page 253]). Thus (use the fact that $N \geq s^{2n}$)

$$\log_s S > nNh(\nu, n) - s^n \log_s N + \frac{1}{2}\log_s 2\pi N - \frac{s^n}{2}\log_s 2\pi s^{-n} N - \frac{s^{2n}\log_s e}{12N}$$

$$> nNh(\nu, n) - \frac{3}{2} N^{1/2}\log_s 2\pi N - \frac{1}{12}\frac{s^{2n}}{N}\log_s e .$$

Proof of Theorem (7.2). It follows from the lemma and Lemma (7.3) and the choice of the sequences $\{n_q\}$ and $\{N_q\}$ that

$$n_p N_p - \sum_{q=1}^{p} (\log_s S_q - n_q N_q h(\nu_q, n_q)) = o(t_p).$$

Thus the inequality

$$\log_s u_p < (\alpha + \varepsilon) t_p - \sum_{q=1}^{p} n_q N_q h(\nu_q, n_q)$$

holds for each $\varepsilon > 0$ and for all sufficiently large p. But $n_q = m_q!$; hence $h(\nu_q, n_q) \geq h(\nu_q)$ for all q. (This is a well-known result from the theory of entropy, see for instance [4, page 49].) Thus

$$\log_s u_p \leq (\alpha - h + \varepsilon) t_p,$$

for all large p. We have shown that for all numbers α, ε $(0 < \alpha < h,\ \varepsilon > 0)$ with $(\alpha - h + \varepsilon) < -\varepsilon$, the inequality $\log_s u_p < -\varepsilon t_p$ holds for all sufficiently large p. This implies that $\sum u_p$ converges. It now follows from the theorem of Eggleston [3, Theorem 4] that

$$\dim R \geq h \equiv \inf_{\nu \in V} h(\nu) .$$

From (4.4), the theorem, and Theorem (7.2), we conclude that

$$\dim G(V, s) = \inf_{\nu \in V} h(\nu) .$$

I am indebted to J. H. B. Kemperman for much guidance.

REFERENCES

1. J. Cigler and G. Helmberg, *Neuere Entwicklungen der Theorie der Gleichverteilung.* Jber. Deutsch. Math. Verein. 64, Abt. 1, (1961), 1-50.
2. H. G. Eggleston, *The fractional dimension of a set defined by decimal properties.* Quart. J. Math., Oxford Ser. 20 (1949), 31-36.
3. ———, *Sets of fractional dimensions which occur in some problems of number theory.* Proc. London Math. Soc. (2) 54 (1952), 42-93.
4. A. I. Khinchin, *Mathematical foundations of information theory.* Dover Publications, New York, 1957.

5. J. C. Oxtoby, *Ergodic sets*. Bull. Amer. Math. Soc. 58 (1952), 116-136.

6. Bodo Volkmann, *Über Hausdorffsche Dimensionen von Mengen, die durch Ziffern-eigenschaften charakterisiert sind.* VI. Math. Z. 68 (1958), 439-449.

7. E. T. Whittaker and G. N. Watson, *A course of modern analysis*. 4th Edition. Cambridge University Press, 1946.

State University College
Geneseo, New York 14454