# MONTGOMERY-SAMELSON COVERINGS ON SPHERES

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## 1. INTRODUCTION

In this note, we study maps f: $M \to N$ from a compact manifold onto a compact manifold. Such a map is called a *Montgomery-Samelson covering* if $f \mid (M - B_f)$ is a covering map onto $N - fB_f$ and $f \mid f^{-1} fB_f$ is a homeomorphism onto $fB_f$, where $B_f$ is the set of points of M at which f is not a local homeomorphism. Furthermore, we assume that $\dim B_f \leq n - 2$ and that the Čech homology groups of $B_f$ are finitely generated. For the rest of this note, f denotes such a map. All spaces, except $B_f$, are manifolds unless exceptions are explicitly noted. $S^n$ denotes the n-sphere, and d $(d > 1)$ is the degree of f. We prove the following results.

THEOREM 1. *If* f: $S^n \to S^n$ *satisfies the requirements above, then* $B_f$ *is an* (n - 2)*-dimensional homology sphere, modulo each prime dividing* d.

This theorem answers a question raised by H. Hopf [5, paragraph 3] and E. Hemmingsen [3, p. 328].

THEOREM 2. *If* f: $M \to S^n$ *is a Montgomery-Samelson covering and* $fB_f$ *is a trivially knotted* p*-sphere in* $S^n$, *then* $p = n - 2$, *the manifold* M *is a topological sphere, and* f *is the* (n - 1)*-fold suspension of a* d*-to-*1 *covering map of* $S^1$ *on* $S^1$.

We adapt to the setting of codimension zero some techniques that P. L. Antonelli devised in his work on Montgomery-Samelson fiberings [1], [2]. The proof of Theorem 1 uses a special homology analogous to that of P. A. Smith [7].

## 2. SPECIAL HOMOLOGY

PROPOSITION. *Let* f: $M \to N$ *be a Montgomery-Samelson covering. Let* p *be some prime dividing* d. *Let* H *denote Čech homology with coefficients in* $Z_p$ *(the integers modulo* p). *Then there exist graded* $Z_p$*-modules* $H^\tau(M)$, $H^\tau(M, B_f)$, $H^\sigma(M)$, *and* $H^\sigma(M, B_f)$ *such that*

(a) *for each* m, *there exist exact sequences*

$$H^\sigma_{m+1}(M, B_f) \to H^\tau_m(M, B_f) \oplus H_m(B_f) \to H_m(M)$$

*and*

$$H^\tau_{m+1}(M, B_f) \to H^\sigma_m(M, B_f) \oplus H_m(B_f) \to H_m(M)\,,$$

*and*

(b) $H^\sigma_m(M, B_f)$ *is the homomorphic image of* $H_m(N, B_f)$.

*Proof.* Part (a) of this theorem is proved in [4]. The homomorphism of part (b) is induced at the chain level in the simplicial case if to each simplex s in $(N, B_f)$,

Received February 5, 1969.
Supported in part by Army Research Office-Durham grant ARO-D-31-124-G892,S2.

Michigan Math. J. 17 (1970).

we assign the chain $\tau(s')$ for some $s'$ in $f^{-1}(s)$. It is usually not an isomorphism, because of the special boundaries. See [6].

## 3. PROOF OF THEOREM 1

It was shown in [4] that under the hypotheses of Theorem 1, we have that $\sum_0^\infty \dim H_i(B_f) \leq 2$, where $\dim H_i(B_f)$ stands for the dimension of the vector space $H_i(B_f)$ over $Z_p$. Since $B_f \neq \emptyset$, it follows that $\dim H_0(B_f) > 0$. Let r be the greatest integer with $H_r(B_f) \neq 0$. By hypothesis, $\dim B_f \leq n - 2$; hence $r \leq n - 2$. Suppose that $r < n - 2$. We infer from the second exact sequence of the proposition that $H^\tau_{r+1}(S^n, B_f) \neq 0$ and from the first that $H^\sigma_{r+2}(S^n, B_f) \neq 0$. By part (b) of the proposition, $H_{r+2}(S^n, fB_f) \neq 0$. Since $r < n - 2$, it follows from the exact sequence for $(S^n, fB_f)$ that $H_{r+1}(B_f) \neq 0$, contrary to the choice of r. Theorem 1 follows.

COROLLARY 1. *If* $n = 4$ *and* $B_f$ *is tamely embedded, then* $B_f = S^2$.

*Proof.* We may assume that f is simplicial [6, Theorem 1]. Then $B_f$ is a 2-manifold, and $X(B_f) = 2$ [3, Theorem 1]. By Theorem 1, $B_f$ is connected; hence $B_f = S^2$.

COROLLARY 2. *If* $g: M^5 \to N^5$ *is a simplicial Montgomery-Samelson covering, then* $B_g$ *is the disjoint union of* 3*-manifolds.*

For a proof, see [3, corollary to Theorem 1].

## 4. PROOF OF THEOREM 2

The space $S^n - S^p = S^n - fB_f$ has the homotopy type of $S^{n-p-1}$ and admits the nontrivial covering $f \mid (M - B_f)$; but this can occur only if $n - p - 1 = 1$. Therefore $p = n - 2$. Now consider $f \mid B_f$. Since this restricted map is a homeomorphism, we have that

$$B_f = fB_f = S^{n-2}.$$

Pick a pair of antipodal points $(p', q')$ in $fB_f$. Let $f^{-1}(p') = p$ and $f^{-1}(q') = q$. Since $fB_f$ is trivially knotted, we may assume that $(p', q')$ is also an antipodal pair in $S^n$. Let $Y'$ be the equatorial sphere in $S^n$ relative to $(p', q')$, and let $M' = f^{-1}(Y')$. Let $F: Y' \times I \to S^n$ be the homotopy between the inclusion of $Y'$ in $S^n$ and the constant map $p'$ obtained by contracting along meridians. Let

$$F_1 = F \mid (Y' - fB_f) \times I \quad \text{and} \quad F_2 = F \mid (Y' \cap fB_f) \times I.$$

The homotopy $F_1$ lifts through f to a homotopy $G_1$ between the inclusion of $M' - B_f$ in $M - B_f$ and the constant map p, and $G_1$ is stationary with $F_1$ by the Covering Homotopy Theorem. Since $f \mid B_f$ is a homeomorphism, $F_2$ can be lifted to a homotopy $G_2$ between the inclusion $M' \cap B_f$ in $B_f$ and the constant map p. Let G be given by $G_1$ on $(M' - B_f) \times I$ and by $G_2$ on $(M' \cap B_f) \times I$. Let U be an open set in $G(M' \times I)$. Then the set

$$G^{-1}(U) = (f \times id)^{-1} \cdot F^{-1} \cdot f(U)$$

is open, because the functions on the right side of the equation are continuous and because f is open. Let H denote the homotopy obtained by contracting $Y'$ to $q'$ along meridians, and let K denote the homotopy that covers H. Each of the sets $G(M' \times I)$ and $K(M' \times I)$ is homeomorphic to a cone over M', and their intersection is M'. Therefore M is homeomorphic to the suspension of M', and f is topologically equivalent to the suspension of $g = f \mid M'$. It is easy to verify that $g: M' \to Y'$ is a d-to-1 Montgomery-Samelson covering of manifolds and that $B_g = B_f \cap Y'$. We know that Y' is an (n - 1)-sphere and $B_f \cap Y'$ is a trivially knotted (n - 3)-sphere. The obvious induction terminates with a d-to-1 covering of $S^1$ by $S^1$. Therefore $f: M \to S^n$ is topologically equivalent to the (n - 1)-fold suspension of a covering of $S^1$ by $S^1$. Then, in particular, M is the (n - 1)-fold suspension of $S^1$; hence M is $S^n$.

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