CONCORDANCE CLASSES OF SPHERE BUNDLES OVER SPHERES

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The purpose of this paper is to provide a proof for a theorem announced in [5] concerning the classification, up to concordance, of differentiable structures on manifolds that are sphere bundles over spheres. This is finer than classification up to diffeomorphism; R. DeSapio [1] has proved results on the latter problem that are interesting to compare with ours.

We recall that a *concordance* between two differentiable structures β and β' on a nonbounded PL (piecewise-linear) manifold K is a differentiable structure γ on the PL manifold $K \times I$ that equals β on $K \times 0$ and β' on $K \times 1$. If we denote the set of equivalence classes under the relation of concordance by C(K), the theorem in question may be stated as follows.

THEOREM. *Let* K *be the total space of an* S^j*-bundle over* S^i *whose characteristic map may be pulled back to an element* α *of* $\pi_{i-1}(SO(j))$. *Then there exists a one-to-one correspondence*

$$C(K) \longleftrightarrow \Gamma_i \oplus A \oplus [\Gamma_{i+j}/\text{image } \tau_\alpha],$$

where A *is a subgroup of* Γ_j. *If* α *can be pulled back to an element* α' *of* $\pi_{i-1}(SO(j-1))$, *then* $A = \Gamma_j \cap (\text{kernel } \tau_{\alpha'})$.

Here Γ_n denotes the group of diffeomorphisms of S^{n-1}, modulo the subgroup consisting of those diffeomorphisms extendable to B^n. It is isomorphic with $C(S^n)$, the operation being connected sum. For $n \geq 5$, $C(S^n)$ is equal to the group Θ_n of (oriented diffeomorphism classes of) differentiable structures on S^n. The group Γ_n is abelian and finite; it vanishes for $n \leq 6$.

The maps $\tau_{\alpha'}\colon \Gamma_j \to \Gamma_{i+j-1}$ and $\tau_\alpha\colon \Gamma_{j+1} \to \Gamma_{i+j}$ are the so-called "Milnor-Munkres-Novikov" twisting homomorphisms. (See [5, p. 189], where the homomorphism

$$\tau\colon \pi_k(SO(m-1)) \otimes \Gamma_m \to \Gamma_{m+k}$$

is defined; in the present paper we denote $\tau(\alpha, x)$ by $\tau_\alpha(x)$.)

Examples. Suppose K is the nontrivial S^j-bundle over S^2 $(j > 1)$; we compare its concordance classes with those of the trivial bundle $S^j \times S^2$. Of course, C(K) is never larger than $C(S^j \times S^2)$, since $\tau_\alpha = 0$ if $\alpha = 0$; and there exist many values of j for which C(K) is strictly smaller, for example, j = 7, 13, 14, 15, and $j \equiv 0$ or $j \equiv 1$ modulo 8. This follows from the fact that if $\alpha(k)$ is the nontrivial element of $\pi_1(SO(k))$ $(k > 2)$, then $\tau_{\alpha(k)}\colon \Gamma_{k+1} \to \Gamma_{k+2}$ is nontrivial for k = 7, 13, 15 and for $k \equiv 0 \pmod 8$. (See J. Levine [4], noting that his $\delta(\sigma, 0; 0, \alpha)$ is just our $\tau_\alpha(\sigma)$.)

As a second example, take K to be an S^j-bundle over S^i having two independent cross-sections (so that α' exists), where j is fairly close to i $(1 \leq i - 3 \leq j \leq i + 1)$;

Received July 18, 1969.

Michigan Math. J. 17 (1970).

we compare the concordance and diffeomorphism classifications in this case. In these dimensions both τ_α and $\tau_{\alpha'}$ vanish [1], so that we obtain for the concordance classes the correspondence

$$C(K) \longleftrightarrow \Gamma_i \oplus \Gamma_j \oplus \Gamma_{i+j}.$$

On the other hand, the oriented diffeomorphism classes are in one-to-one correspondence with a quotient merely of Γ_{i+j}; this quotient is equal to Γ_{i+j} when the bundle is trivial [1].

1. A GEOMETRIC APPROACH TO THE THEOREM

An obvious question arises: if K is a manifold satisfying the hypotheses of the theorem, how does one describe the differentiable structure on K corresponding to a prescribed element of

$$(*) \qquad \Gamma_i \oplus (\Gamma_j \cap \text{kernel } \tau_{\alpha'}) \oplus (\Gamma_{i+j}/\text{image } \tau_\alpha)\ ?$$

The answer is as follows: Choose a fixed cross-section and a fixed fibre. Let Σ^i, Σ^j, and Σ^{i+j} be differentiable spheres representing elements of Γ_i, Γ_j, and $\Gamma_{i+j}/\text{image } \tau_\alpha$. Impose, if you can, a differentiable structure β on K under which the distinguished cross-section inherits the structure Σ^i and the distinguished fibre inherits the structure Σ^j; then assign to $(\Sigma^i, \Sigma^j, \Sigma^{i+j})$ the connected sum $K_\beta \# \Sigma^{i+j}$. The structure β exists for every Σ^i. The set of Σ^j for which β exists includes $\Gamma_j \cap (\text{kernel } \tau_{\alpha'})$, provided α' exists. (A short geometric argument is involved here.) The concordance class of the result is well-defined [6].

Our original proof of the theorem proceeded along the lines that this description suggests. We used the theorem [5] stating that there exists an injection

$$C(K) \to \textstyle\sum_p H^p(K;\ \Gamma_p)/\text{images } \Lambda^k,$$

and we computed the latter group to be

$$H^i(K;\ \Gamma_i) \oplus H^j(K;\ \Gamma_j) \oplus (H^{i+j}(K;\ \Gamma_{i+j})/\text{image } \Lambda^i),$$

which is isomorphic with

$$(**) \qquad \Gamma_i \oplus \Gamma_j \oplus \Gamma_{i+j}/\text{image } \tau_\alpha,$$

by 2.3 of [6]. Then we verified that the injection of C(K) into this group could in fact be described by the geometric construction of the preceding paragraph. It follows at once that if α' exists, C(K) is at least as big as the group (*) and no bigger than the group (**). To show the existence of the group A in general and to show the equality of A with the kernel of $\tau_{\alpha'}$ if α' exists is more difficult; it requires a rather long and messy geometric argument.

There is, however, an alternate method of proof, which we were unaware of when we wrote [5], one that uses homotopy-theoretic techniques instead of geometric ones. Here we present the latter proof.

2. A HOMOTOPY-THEORETIC APPROACH

Recall that M. Hirsch and B. Mazur have announced [2] the construction of a homotopy-associative, homotopy-commutative h-space Γ such that for each non-bounded PL manifold K and each differentiable structure β on K, there exists a one-to-one correspondence between C(K) and [K, Γ] (the set of homotopy classes of maps of K into Γ) taking β to the constant map. The correspondence is natural with respect to inclusions of open subsets of K. In the particular case where $K = S^n$, the correspondence

$$C(S^n) \longleftrightarrow [S^n, \Gamma] = \pi_n(\Gamma)$$

preserves the group operation if we let the usual differentiable structure on S^n correspond to the constant map.

We use this result to prove our theorem; the proof reduces to computing [K, Γ]. We have the following purely homotopy-theoretic lemma (proved in Section 3).

LEMMA. *Assume* K *is an* S^j*-bundle over* S^i, *as in the preceding theorem. Then there exists a one-to-one correspondence*

$$[K, \Gamma] \longleftrightarrow \pi_i(\Gamma) \oplus A \oplus \pi_{i+j}(\Gamma)/\text{image } J_\alpha ,$$

where A *is a subgroup of* $\pi_j(\Gamma)$. *If* α' *exists, then* $A = \pi_j(\Gamma) \cap (\text{kernel } J_{\alpha'})$.

Here $J_\alpha \colon \pi_{j+1}(\Gamma) \to \pi_{i+j}(\Gamma)$ is obtained from the J-homomorphism of homotopy theory by defining $J_\alpha(x)$ to be $J(i(\alpha)) \circ x$, the composition of the image of α under the maps

$$\pi_{i-1}(SO(j)) \xrightarrow{i} \pi_{i-1}(SO(j+1)) \xrightarrow{J} \pi_{i+j}(S^{j+1})$$

(where i is inclusion) with the element x of $\pi_{j+1}(\Gamma)$. It is easy to see that $J_\alpha(x)$ is bilinear in α and x. (Note that $J(\alpha'') \circ x$ is defined for α'' in $\pi_{i-1}(SO(j+1))$, but bilinearity fails.)

Since $\pi_k(\Gamma)$ and Γ_k are isomorphic, both being isomorphic with $C(S^k)$, we need only to identify J_α with τ_α in order to obtain our theorem.

We stated this identification as obvious at the end of our expository paper [5]; now we are not so sure. We expect that it can be proved directly, using the techniques of Hirsch and Mazur; but since their work has not yet appeared, we content ourselves here with an indirect proof in Section 4 that the images of J_α and τ_α have the same order. For this purpose, we consider the subset of C(K) consisting of those structures that differ from the standard one only within a combinatorial ball. By our smoothing theory [6], we know that this subset is in one-to-one correspondence with $\Gamma_{i+j}/\text{image } \tau_\alpha$; from the lemma of Section 3, we know that this subset is in one-to-one correspondence with $\pi_{i+j}(\Gamma)/\text{image } J_\alpha$. Our theorem follows.

3. PROOF OF THE LEMMA

We prove the lemma in the following form.

LEMMA. *Let* Γ *be a homotopy-associative, homotopy-commutative, path-connected* h*-space. For each* K, *give* [K, Γ] *a group structure, using the multiplication in* Γ *to define the operation. If* K *is an* S^j*-bundle over* S^i *whose* *characteristic map can be pulled back to an element* α *in* $\pi_{i-1}(SO(j))$, *then there exist an abelian group* G *and exact sequences*

$$0 \longrightarrow G \xrightarrow{i} [K, \Gamma] \xrightarrow{\lambda} \pi_i(\Gamma) \longrightarrow 0,$$

$$\pi_{j+1}(\Gamma) \xrightarrow{J_\alpha} \pi_{i+j}(\Gamma) \xrightarrow{\mu} G \xrightarrow{\nu} \pi_j(\Gamma),$$

where the first sequence splits. If α *can be pulled back to an element* α' *in* $\pi_{i-1}(SO(j-1))$, *then one may adjoin*

$$\pi_j(\Gamma) \xrightarrow{J_{\alpha'}} \pi_{i+j-1}(\Gamma)$$

to the second sequence and retain exactness.

The image of $i\mu$ *consists of the homotopy classes of those maps that are constant outside a combinatorial ball.*

Proof. (1) *The homomorphism* J_α. We give the definition of J_α in a more useful form as follows. Let $\psi: S^{i-1} \to SO(j)$ represent the element α of $\pi_{i-1}(SO(j))$. Let

$$h: (B^{j+1}, S^j) \to (\Gamma, e)$$

represent the element x of $\pi_{j+1}(\Gamma)$. Then $-J_\alpha(x)$ is represented by the map

$$\phi: \partial(B^i \times B^{j+1}) \to \Gamma$$

defined by the equations

$$\phi(x_1, z) = h(\psi(x_1)\cdot z) \qquad (x_1 \in \partial B^i),$$

$$\phi(x, z_1) = e \qquad (z_1 \in \partial B^j).$$

(Here $\psi(x_1)$ acts on the first j coordinates in R^{j+1}.) To see this, compare ϕ with the original definition of the J-homomorphism in [7], where it is denoted by $H_{m,n}$. The minus sign occurs for the reason that vertical projection of B^m onto the lower hemisphere of S^m carries the standard orientation of B^m onto the opposite of the one induced from the standard orientation of S^m.

(2) *Notation.* Because K has a cross-section, we may consider it as the space $B^i \times B^j$, modulo the identifications

$$(x, y_1) = (x, y_2) \qquad (y_1, y_2 \in \partial B^j) \text{ and}$$

$$(x_1, \psi(x_1)^{-1}\cdot y) = (x_2, \psi(x_2)^{-1}\cdot y) \qquad (x_1, x_2 \in \partial B^i).$$

Let $\rho: B^i \times B^j \to K$ be the identification map. Choose base points x_0 and y_0 in ∂B^i and ∂B^j, respectively; let $S^i = \rho(B^i \times y_0)$ and $S^j = \rho(x_0 \times B^j)$ be the standard cross-section and fibre, respectively. Assume ψ sends the base point x_0 into the identity of SO(j). Let $\pi: K \to S^i$ be bundle projection. In other words, we have the diagram

$$\begin{array}{ccc} B^i \times B^j & \xrightarrow{\rho} & K \\ \downarrow & & \downarrow \pi \\ B^i & \xrightarrow{\rho} & S^i . \end{array}$$

(3) *Construction of the exact sequences.* Define G to be the subgroup of $[K, \Gamma]$ consisting of homotopy classes of maps $f: K \to \Gamma$ that are constant on S^i. Let the map i be inclusion; define λ by the equation $\lambda[f] = [f \mid S^i]$. The first sequence is clearly exact. If we define $\kappa: \pi_i(\Gamma) \to [K, \Gamma]$ by the equation $\kappa[g] = [g\pi]$, where $g: S^i \to \Gamma$, then $\lambda\kappa$ is the identity; hence the first sequence splits.

Define μ by the equation $\mu[g] = [f]$, where

$$g: (B^i \times B^j, \partial(B^i \times B^j)) \to (\Gamma, e)$$

and $f: K \to \Gamma$ is the map induced by g. Define $\nu[f] = [f \mid S^j]$. Clearly, exactness holds at G.

Note that the image of $i\mu$ consists of the homotopy classes of those maps f for which $f \mid S^i \cup S^j$ is constant. Every such homotopy class is represented by a map that is constant outside a combinatorial ball, and conversely.

(4) *Exactness at* $\pi_{i+j}(\Gamma)$. Let

$$g: (B^i \times B^j, \partial(B^i \times B^j)) \to (\Gamma, e) \quad \text{and} \quad h: (B^j \times I, \partial(B^j \times I)) \to (\Gamma, e)$$

represent the general elements of $\pi_{i+j}(\Gamma)$ and $\pi_{j+1}(\Gamma)$, respectively. Define a map

$$\Phi: \partial(B^i \times B^j \times I) \to \Gamma,$$

depending on g and h, by the equations

$$\Phi(x, y, 1) = g(x, y), \qquad \Phi(x, y, 0) = e,$$

$$\Phi(x, y_1, t) = e \quad (y_1 \in \partial B^j), \qquad \Phi(x_1, y, t) = h(\psi(x_1)\cdot y, t) \quad (x_1 \in \partial B^i).$$

We show (a) that $[\Phi] = [g] - J_\alpha[h]$, and (b) that for a prescribed g, $\mu[g] = 0$ if and only if, for some choice of h, the map Φ determined by g and h is homotopic to zero.

(a) For a prescribed Φ, let

$$\Phi_1: \partial(B^i \times B^j \times I) \to \Gamma$$

equal Φ on $B^i \times B^j \times 1$ and equal e elsewhere; let Φ_2 equal e on $B^i \times B^j \times 1$ and equal Φ elsewhere. Then $[\Phi] = [\Phi_1] + [\Phi_2]$. Now Φ_1 and g represent the same element of $\pi_{i+j}(\Gamma)$, because Φ_1 equals g on $B^i \times B^j \times 1$ and equals e elsewhere. On the other hand, Φ_2 represents the element $-J_\alpha[h]$ of $\pi_{i+j}(\Gamma)$, for if we take the standard homeomorphism $\theta: (x, y, t) \to (x, z)$ of $B^i \times B^j \times I$ with $B^i \times B^{j+1}$, then the map

$$\theta\Phi_2\theta^{-1}: \partial(B^i \times B^{j+1}) \to \Gamma$$

is precisely the map ϕ defined in part (1) of this proof. (Note that $\theta(x_1, \psi(x_1)\cdot y, t)$ equals $(x_1, \psi(x_1)\cdot z)$, because $\psi(x_1)$ acts only on the first j coordinates of z.)

(b) Now $[\Phi] = 0$ if and only if Φ is extendable to $B^i \times B^j \times I$. If Φ is extendable, the extension induces a map $F: K \times I \to \Gamma$ that is a homotopy between the map $f: K \to \Gamma$ induced by g and the constant map e, so that $\mu[g] = [f] = 0$.

Conversely, if $\mu[g] = 0$, there exists a homotopy F between the map f induced by g and the constant map e. We shall construct a homotopy F' between f and e that leaves S^i fixed at e. It follows at once that the map Φ determined by g and h will be extendable, provided we choose $h(y, t) = F'(\rho(x_0, y), t)$; for then $F'(\rho(x, y), t)$ will be the desired extension.

Let $a = F \mid S^i \times I$; then $a: (S^i \times I, S^i \times \partial I) \to (\Gamma, e)$. Choose a map

$$b: (S^i \times I, S^i \times \partial I) \to (\Gamma, e)$$

such that $a \cdot b: (S^i \times I, S^i \times \partial I) \to (\Gamma, e \cdot e)$ is homotopic to a constant. (The map a induces a map $S^{i+1} \to \Gamma$ carrying the north and south poles to e; choose b to induce a homotopy inverse for a.) Define B as the composite

$$K \times I \xrightarrow{\pi \times id} S^i \times I \xrightarrow{b} (\Gamma, e);$$

then $F \cdot B$ is a homotopy between $f \cdot e$ and $e \cdot e$ whose restriction to $S^i \times I$ is homotopically trivial. The homotopy extension theorem gives us the desired homotopy F'.

(5) *Exactness at* $\pi_j(\Gamma)$. Assume α can be pulled back to α' in $\pi_{i-1}(SO(j-1))$. Let ψ, h, and ϕ be as in part (1) above, except that j is replaced by j - 1 throughout. If ψ represents α', then ϕ represents $-J_{\alpha'}[h]$; this element vanishes if and only if ϕ is extendable to $B^i \times B^j$. If ϕ is extendable, then the extension induces a map $f: K \to \Gamma$ that is constant on S^i, and $\nu[f] = [f \mid S^j] = [h]$. Conversely, if $[h] = \nu[f]$ for some f, then $f\rho$ is an extension of ϕ.

4. THE RELATION OF τ_α AND J_α

Definition. If K is a nonbounded PL n-manifold with differentiable structure β, let $C'_\beta(K)$ denote the subset of C(K) consisting of those concordance classes containing differentiable structures that are equal to β in a neighborhood of the complement of a combinatorial ball. We interpret $C'_\beta(K)$ in two ways.

First interpretation. Choose the Hirsch-Mazur correspondence $C(K) \longleftrightarrow [K, \Gamma]$ so that β corresponds to the constant map. Then $C'_\beta(K)$ corresponds to the homotopy classes of those maps that are constant outside a combinatorial ball.

Let β' agree with β in a neighborhood U of K - D, for some combinatorial ball D in K; let $f: K \to \Gamma$ be the map corresponding to β'. By naturality of the correspondence under inclusions of open sets, $f \mid U$ is homotopic to a constant, so that f is homotopic to a map constant outside D.

Conversely, if f is constant outside D, let U equal K minus an interior point of D. Since $f \mid U$ is homotopically trivial, $\beta \mid U$ and $\beta' \mid U$ are concordant. The concordance-extension theorem of Hirsch and Mazur implies that β' is concordant to a structure β'' that equals β in a neighborhood of K - D.

Second interpretation. In [6], we showed how the formation of the connected sum $K_\beta \# \Sigma^n$ of K_β with a differentiable sphere Σ^n defined an *action* of Γ_n on C(K). The concordance class of $K_\beta \# \Sigma^n$ can be represented by a differentiable structure that equals β outside a combinatorial ball, and conversely. We used $I_c(K_\beta)$ to denote the subgroup of Γ_n consisting of those elements that act trivially on the concordance class of β; we called $I_c(K_\beta)$ the *concordance inertia group* of K_β. The one-to-one correspondence

$$C'_\beta(K) \longleftrightarrow \Gamma_n/I_c(K_\beta)$$

follows at once.

LEMMA. *Let* K_β *be the* S^i*-bundle over* S^i *of our theorem, with its usual differentiable structure. Then there exist one-to-one correspondences*

$$\pi_{i+j}(\Gamma)/\text{image } J_\alpha \longleftrightarrow C'_\beta(K) \longleftrightarrow \Gamma_{i+j}/\text{image } \tau_\alpha .$$

Proof. By the lemma of Section 3 and the first interpretation of $C'_\beta(K)$, we have a one-to-one correspondence

$$C'_\beta(K) \longleftrightarrow \text{image } i\mu ,$$

and an isomorphism

$$\text{image } i\mu \cong \pi_{i+j}(\Gamma)/\text{image } J_\alpha .$$

On the other hand, by Theorem 2.5 of [6], we have the equation $I_c(K_\beta) = \text{image } \tau_\alpha$; now the second interpretation of $C'_\beta(K)$ gives us the one-to-one correspondence

$$C'_\beta(K) \longleftrightarrow \Gamma_{i+j}/\text{image } \tau_\alpha .$$

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