PRIMES IN ARITHMETIC PROGRESSIONS

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The purpose of this paper is to investigate the asymptotic size of the quantity

$$\sum_{q \le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \psi(x;\, q,\, a) - \frac{x}{\phi(q)} \right|^2, \tag{1}$$

where

$$\psi(x;\, q,\, a) = \sum_{\substack{n \le x \\ n \equiv a(q)}} \Lambda(n).$$

In recent years, a number of mathematicians have used the large sieve to obtain upper bounds for (1) when Q is slightly smaller than x. M. B. Barban [1] seems to have been the first, and he was closely followed by H. Davenport and H. Halberstam [6], who showed that (1) is $\ll Qx(\log x)^5$ (Vinogradov's notation) for $Q \le x$. Later, P. X. Gallagher [8] introduced some simplifications and made more precise estimates to show that (1) is $\ll Qx \log x$, and R. J. Wilson used the large sieve in an algebraic number field to obtain similar results in algebraic number fields [12]. In this paper we show, *without using the large sieve,* that these bounds may be replaced by an asymptotic equality with an explicit error term.

THEOREM. *Let* A *be fixed* ($A > 0$). *For* $Q \le x$,

$$\sum_{q \le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \psi(x;\, q,\, a) - \frac{x}{\phi(q)} \right|^2 = Qx \log x + O\left(Qx \log \frac{2x}{Q}\right) + O(x^2 (\log x)^{-A}), \tag{2}$$

and for $Q \ge x$,

$$\sum_{q \le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \psi(x;\, q,\, a) - \frac{x}{\phi(q)} \right|^2 = Qx \log x - \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} x^2 \log \frac{Q}{x} - Qx + A_1 x^2 + O(Qx (\log x)^{-A}). \tag{3}$$

The first error term in (2) may no doubt be reduced, but it appears that the error is genuinely $\gg Qx \log \log x/Q$. Halberstam has pointed out that in an obscure journal Barban [2] asserted (3) in the case $x = Q$. However, he seems not to have indicated what lines the proof was to take, and it may now be impossible ever to determine what he had in mind.

We note that our method suggests that perhaps

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$$\sum_{q \le x^{1/2}} \phi(q) \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \psi(x; q, a) - \frac{x}{\phi(q)} \right|^2 \ll x^2 (\log x)^B \tag{4}$$

for some fixed $B > 0$. Indeed, it may even be possible to prove (4) under the assumption of the generalized Riemann hypothesis. The need for this hypothesis is suggested by the fact that (4) implies that neither the ζ-function nor any L-function can have a zero ρ with $\Re \rho > 3/4$. Professor Halberstam has remarked that the relation

$$\sum_{p \le x^{1/2}} p \sum_{a=1}^{p-1} \left| \psi(x; p, a) - \frac{x}{\phi(p)} \right|^2 \ll x^2 \log x \tag{5}$$

already follows easily from the large sieve. This is surprising, since (4) does not have the appearance of being much stronger than (5). One should note that $\psi(x; q, a)$ counts primes from a set of only x/q elements, and that a probabilist would tell us that at least "usually" we should therefore expect an error of $(x/q)^{\frac{1}{2}+\varepsilon}$. We see that (4) and (5) are statements of this type. Halberstam conjectured a deeper statement of this sort; a strong form of his conjecture is that for each fixed $A > 0$,

$$\sum_{q \le Q} q \max_{\substack{a \\ (a,q)=1}} \left| \psi(x; q, a) - \frac{x}{\phi(q)} \right|^2 \ll \frac{x^2}{(\log x)^A} \tag{6}$$

provided $Q \le x(\log x)^{-B}$ $(B = B(A))$. Bombieri's theorem [3] (see also [5], [9]) asserts that (6) is true provided $Q \le x^{1/2}(\log x)^{-B}$ $(B = B(A))$, but that much would be true even if the error terms were usually $x^{1/2}(\log x)^2$ in size. A much weaker form of Halberstam's conjecture states that

$$\sum_{q \le x^{1-\varepsilon}} q \left| \psi(x; q, 2) - \frac{x}{\phi(q)} \right|^2 \ll \frac{x^2}{(\log x)^{20}}; \tag{7}$$

even this would have implications in the twin-prime problem. At the moment (4) and (5), insofar as they are true or seem plausible, form our only evidence in favor of (6) and (7).

As we have said, the proof of our theorem does not use the large sieve. It does however appeal to a deep result of A. F. Lavrik [10] on *twin primes on average*, based on the work of N. G. Čudakov [4] (see also [7]), which in turn was based on the work of I. M. Vinogradov. We give a formulation of Lavrik's result in Lemma 1.

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In what follows, we let

$$\mathfrak{S} = 2 \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2} \right), \tag{8}$$

(9) $$\psi_2(x, k) = \sum_{k<n\leq x} \Lambda(n)\Lambda(n-k) \qquad (k>0),$$

and

(10) $$E(x, k) = \psi_2(x, 2k) - \mathfrak{S}(x-2k) \prod_{\substack{p\mid k\\ p>2}} \frac{p-1}{p-2}.$$

The following proposition is basic to our proof.

LEMMA 1 (Lavrik). *Let* $\mathfrak{S}$, $\psi_2(x, k)$, *and* $E(x, k)$ *be defined as in* (8), (9), *and* (10). *Then, for each fixed* $B > 0$,

(11) $$\sum_{k=1}^{x/2} (E(x, k))^2 \ll \frac{x^3}{(\log x)^B}.$$

We shall also require the following two results from elementary number theory. Since our proof is standard, we shall omit some of the more tedious details.

LEMMA 2. *Let* $\mathfrak{S}$ *be defined as in* (8). *For* $y \geq 2$,

(12) $$\sum_{m\leq y} \prod_{\substack{p\mid m\\ (p,2r)=1}} \frac{p-1}{p-2} = \frac{2y}{\mathfrak{S}} \prod_{\substack{p\mid r\\ p>2}} \left(1 - \frac{1}{(p-1)^2}\right) + O(\log y)$$

and

(13) $$\sum_{m\leq y} m \prod_{\substack{p\mid m\\ (p,2r)=1}} \frac{p-1}{p-2} = \frac{y^2}{\mathfrak{S}} \prod_{\substack{p\mid r\\ p>2}} \left(1 - \frac{1}{(p-1)^2}\right) + O(y\log y);$$

the error terms are uniform in the positive integer r.

Our proof follows in spirit the proof of the well-known result

(14) $$\sum_{n\leq x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x\log x),$$

but some care must be given to ensure the uniformity of the error term. It is curious that while the error term in (14) may be replaced [11, p. 114] by $O(x(\log x)^{2/3}(\log\log x)^{4/3})$, deeper arguments show that the error-term in (12) cannot be made smaller.

We now prove (12). Let

$$f_r(m) = \begin{cases} \mu^2(m) \prod_{p\mid m} (p-2)^{-1} & \text{if } (m, 2r) = 1, \\ 0 & \text{if } (m, 2r) > 1. \end{cases}$$

Then the left-hand side in (12) is

$$\sum_{m\le y}\sum_{d|m} f_r(d) = \sum_{d\le y} f_r(d) \sum_{\substack{m\le y\\ d|m}} 1$$

$$= y\sum_{d\le y}\frac{f_r(d)}{d} + O\Big(\sum_{d\le y} f_1(d)\Big)$$

$$= y\sum_{d=1}^{\infty}\frac{f_r(d)}{d} + O\Big(y\sum_{d\ge y}\frac{f_1(d)}{d}\Big) + O\Big(\sum_{d\le y} f_1(d)\Big)$$

$$= \frac{2y}{\mathfrak{S}}\prod_{\substack{p|r\\ p>2}}\Big(1-\frac{1}{(p-1)^2}\Big) + O(\log y).$$

Now (13) follows from (12) by partial summation.

LEMMA 3. *For* $x^{1/2}\le Q\le x$,

$$\sum_{q\le Q}\sum_{\substack{k\le x/q\\ 2|qk}}(x-qk)\prod_{\substack{p|qk\\ p>2}}\frac{p-1}{p-2} = \frac{x^2}{2\mathfrak{S}}\sum_{q\le Q}\frac{1}{\phi(q)} + O\Big(Qx\log\frac{2x}{Q}\Big), \tag{15}$$

and for $Q\ge x$,

$$\sum_{q\le Q}\sum_{\substack{k\le x/q\\ 2|qk}}(x-qk)\prod_{\substack{p|qk\\ p>2}}\frac{p-1}{p-2} = \frac{\zeta(2)\zeta(3)}{2\mathfrak{S}\,\zeta(6)}x^2\log x + A_2x^2 + O(x^{3/2}\log x). \tag{16}$$

On the left-hand side of (15) and (16), the condition $2\mid qk$ may be expressed as $2\mid k$ or $2\mid q$; therefore the expression is

$$2F\Big(Q,\frac{x}{2}\Big) + 2F\Big(\frac{Q}{2},\frac{x}{2}\Big) - 4F\Big(\frac{Q}{2},\frac{x}{4}\Big), \tag{17}$$

where

$$F(U,y) = \sum_{u\le U}\sum_{v\le y/u}(y-uv)\prod_{\substack{p|uv\\ p>2}}\frac{p-1}{p-2}.$$

Now, for $U\le y$,

$$F(U,y) = \sum_{u\le U}\prod_{\substack{p|u\\ p>2}}\frac{p-1}{p-2}\sum_{v\le y/u}(y-uv)\prod_{\substack{p|v\\ (p,2v)=1}}\frac{p-1}{p-2};$$

we appeal to Lemma 2 and simplify, to find that this is equal to

$$\frac{y^2}{\mathfrak{S}} \sum_{u \le U} \frac{1}{u} \prod_{\substack{p|u \\ p>2}} \frac{p}{p-1} + O\left(y \sum_{u \le U} \left(\log \frac{y}{u}\right) \prod_{\substack{p|u \\ p>2}} \frac{p-1}{p-2} \right)$$

$$= \frac{y^2}{\mathfrak{S}} \sum_{u \le U} \frac{1}{\phi^*(u)} + O\left(yU \log \frac{2y}{U} \right),$$

where

$$\phi^*(u) = \begin{cases} \phi(u) & \text{if } r \text{ is odd,} \\ 2\phi(u) & \text{if } r \text{ is even.} \end{cases}$$

We can now weaken the condition $U \le y$ to $U \le 2y$, because the difference may be absorbed into the error-term. Taking this result with (13) and the observation that $\phi^*(u) = \phi(2u)$, we have (15). More careful estimates of $F(U, y)$ can be made along the lines of the elementary treatment of $\sum_{n \le x} d(n)$, and all error-terms are small, except in estimating

$$y^2 \sum_{u \le y/U} \frac{1}{\phi^*(u)} .$$

One cannot hope to estimate this with an error less than $yU \log \log y/U$. Hence the error in (15) cannot be as small as $O(Qx)$ if $Q = o(x)$.

If $U \ge y$, then $F(U, y)$ does not depend on U, and in fact

$$F(U, y) = 2 \sum_{u \le y^{1/2}} \prod_{\substack{p|u \\ p>2}} \frac{p-1}{p-2} \sum_{v \le y/u} (y - uv) \prod_{\substack{p|v \\ (p,2u)=1}} \frac{p-1}{p-2}$$

$$- \sum_{u \le y^{1/2}} \prod_{\substack{p|u \\ p>2}} \frac{p-1}{p-2} \sum_{v \le y^{1/2}} (y - uv) \prod_{\substack{p|v \\ (p,2v)=1}} \frac{p-1}{p-2},$$

and by an appeal to Lemma 2 we find that this is

$$\frac{2y^2}{\mathfrak{S}} \sum_{u \le y^{1/2}} \frac{1}{\phi^*(u)} - \frac{2y^{3/2}}{\mathfrak{S}} \sum_{u \le y^{1/2}} \prod_{\substack{p|u \\ p>2}} \frac{p}{p-1} + \frac{y}{\mathfrak{S}} \sum_{u \le y^{1/2}} u \prod_{\substack{p|u \\ p>2}} \frac{p}{p-1} + O(y^{3/2} \log y).$$

By results similar to Lemma 2, this is equal to

$$\frac{y^2 \log y}{\mathfrak{S}} \prod_{p>2} \left(1 + \frac{1}{p(p-1)}\right) + A_3 y^2 + O(y^{3/2} \log y).$$

This, with (17), gives (16).

We now prove our theorem. First we note that

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\psi(x;q,a) - \frac{x}{\phi(q)}\right|^2 = \sum_{\substack{a=1\\(a,q)=1}}^{q} |\psi(x;q,a)|^2 - \frac{x}{\phi(q)}\Bigl(2\sum_{\substack{n\le x\\(n,q)=1}} \Lambda(n) - x\Bigr)$$

$$= \sum_{\substack{n_1\le x\\ n_1\equiv n_2(q)\\(n_1n_2,q)=1}} \sum_{n_2\le x} \Lambda(n_1)\Lambda(n_2) - \frac{x^2}{\phi(q)} + O\Bigl(\frac{x^2}{\phi(q)(\log x)^B}\Bigr) + O\Bigl(\frac{x\log q}{\phi(q)}\Bigr).$$

If in the double summation we drop the condition $(n_1 n_2, q) = 1$, we increase the sum by an amount that is $\ll xq^{-1}\log x$. Thus the left-hand side of (2) or (3) is

$$Q\sum_{n\le x}(\Lambda(n))^2 + 2\sum_{q\le Q}\sum_{\substack{n_1<n_2\le x\\ n_1\equiv n_2(q)}} \Lambda(n_1)\Lambda(n_2) - x^2\sum_{q\le Q}\frac{1}{\phi(q)} \tag{18}$$

$$+ O\Bigl(\frac{x^2\log Q}{(\log x)^B}\Bigr) + O(x(\log Q)^2).$$

Now

$$\sum_{n\le x}(\Lambda(n))^2 = x\log x - x + O\Bigl(\frac{x}{(\log x)^A}\Bigr), \tag{19}$$

by the prime-number theorem. Also,

$$2\sum_{q\le Q}\sum_{\substack{n_1<n_2\le x\\ n_1\equiv n_2(q)}} \Lambda(n_1)\Lambda(n_2) = 2\sum_{q\le Q}\sum_{k\le x/q}\psi_2(x, qk).$$

If r is odd, then $\psi_2(x,r) \ll (\log x)^2$; therefore odd values of qk above contribute no more than $\ll x(\log x)^3$. By (10), this is equal to

$$2\mathfrak{S}\sum_{q\le Q}\sum_{\substack{qk\le x\\ 2|qk}}(x-qk)\prod_{\substack{p|qk\\ p>2}}\frac{p-1}{p-2} + O\Bigl(\sum_{2m\le x} d(2m)\,|E(x,m)|\Bigr) + O(x(\log x)^3). \tag{20}$$

Here, by Cauchy's inequality, the first error-term is

$$\ll \Bigl(\sum_{2m\le x} d^2(m)\Bigr)^{1/2}\Bigl(\sum_{2m\le x}(E(x,m))^2\Bigr)^{1/2}.$$

The first factor is $\ll x^{1/2}(\log x)^{3/2}$, and the second is $\ll x^{3/2}(\log x)^{-B/2}$, by Lemma 1. We take $B = 2A + 3$, so that this error is $\ll x^2(\log x)^{-A}$. To prove (2), we have only to combine (18), (19), (20), and (15). As for (3), we take (18), (19), and (20) with (16) and the fact that

$$\sum_{m\le M}\frac{1}{\phi(m)} = \frac{\zeta(2)\,\zeta(3)}{\zeta(6)}\log M + A_4 + O\Bigl(\frac{\log M}{M}\Bigr).$$

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