A COUNTEREXAMPLE TO A CONJECTURE IN SECOND-ORDER LINEAR EQUATIONS

Herbert Arthur DeKleine

Consider the differential equation

$$u'' + a(t)u = 0, \tag{1}$$

where $a(t)$ is a positive, nondecreasing, unbounded function in $C'[T, \infty)$. It is well known that the hypotheses on $a(t)$ do not imply that every solution of (1) satisfies the condition

$$u(t) \to 0 \quad \text{as } t \to \infty. \tag{2}$$

L. A. Gusarov [3] has shown that under the additional hypothesis that $a'(t)$ is of bounded variation on $[T, \infty)$, the solutions of (1) satisfy condition (2). Under these assumptions, $a'(t)$ has a finite, nonnegative limit as $t \to \infty$. A. Meir, D. Willett, and J. S. W. Wong [4] have proved the following result.

THEOREM 1. *If there exists a positive function* $p(t) \in C'[0, \infty)$ *such that*

$$\int_0^\infty \frac{dt}{p(t)} = +\infty, \qquad \liminf_{t\to\infty} \frac{p'(t)}{p(t)a^{1/2}(t)} \geq 0, \qquad \textit{and } \liminf_{t\to\infty} \frac{a'(t)p(t)}{a(t)} > 0,$$

then the solutions of (1) *satisfy condition* (2).

From this result it follows that if $a'(t)$ is ultimately bounded and bounded away from zero, then all solutions of (1) satisfy (2). The following question presents itself: does the condition that $a'(t) \to 0$ as $t \to \infty$ (or that $\limsup a'(t) < \infty$) imply that condition (2) holds for all solutions of (1)? Meir, Willett, and Wong [4] conjectured that if in Theorem 1 the last condition is replaced by the condition

$$\lim_{t\to\infty} a'(t)p(t)/a(t) = 0,$$

then the conclusion remains valid. If this conjecture were true, we could answer our question in the affirmative (simply set $p(t) \equiv 1$). However, the following theorem shows that the conjecture is false.

THEOREM 2. *For each* $\beta > 0$, *there exists a positive function* $a(t) \in C^\infty[0, \infty)$ *such that* $a(t) \to \infty$, $a'(t) \geq 0$, $a'(t) = o(\log^{-\beta} t)$, *and such that at least one solution* $u(t)$ *of* (1) *satisfies the condition* $\limsup_{t\to\infty} |u(t)| > 0$.

Without loss of generality, we replace the condition $a'(t) = o(\log^{-\beta} t)$ by $a'(t) = O(\log^{-m} t)$, where m is an integer $(m > \beta)$. The proof is based on a method

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used by A. S. Galbraith, E. J. McShane, G. B. Parrish [2], and D. Willett [6]. The following lemma, which was established by Willett [6], will be used in the proof of Theorem 2.

LEMMA 1. *Let* u(t) *be a solution of* (1), *and let* μ *be a positive number such that* $a(t) \geq \mu^2$ *for all* $t \in [0, \infty)$. *Then* $u'(t)$ *has at least one zero in each interval of length* $2\pi/\mu$.

Proof of Theorem 2. Consider the functions $\psi(t)$ and $\rho(t)$ defined by

$$\psi(t) = \begin{cases} \exp(1 - t^{-2}) & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases}$$

$$\rho(t) = \psi[1 - \psi(1 - t)].$$

Clearly, $\rho(t)$ is a nondecreasing C^∞-function with values in [0, 1], and it satisfies the conditions $\rho(t) = 0$ $(t \leq 0)$ and $\rho(t) = 1$ $(t \geq 1)$.

Let $t_1 = 0$, $s_1 = 1/2$, $\alpha_0 = 16\pi^2$, and $\alpha_1 = 16\pi^2 + 1$. Define $a_1(t)$ by the condition $a_1(t) = \alpha_0 + (\alpha_1 - \alpha_0)\rho(2t)$. Let $u_1(t)$ denote the unique solution of the initial-value problem $u_1'' + a_1(t)u_1 = 0$, $u_1(0) = 1$, and $u_1'(0) = 0$. By Lemma 1, there exists a point t_2 $(1/2 \leq t_2 < 1)$ such that $u_1'(t_2) = 0$. The following construction is inductive. We choose a sequence $\{\alpha_n\}$, a sequence $0 = t_1 < s_1 < t_2 < s_2 < \cdots$, and a set of functions $u_n(t)$ $(n = 1, 2, \cdots)$ such that

$$\alpha_n = \alpha_{n-1} + n^{-1} = 16\pi^2 + \sum_{k=1}^{n} k^{-1},$$

$$s_n - t_n = \begin{cases} 1/2 & \text{if } n = 1, \\ \min[1/2, n^{-1}\log^m n] & \text{if } n \geq 2, \end{cases}$$

$$n - 3/2 \leq t_n \leq n - 1,$$

$$a_n(t) = \alpha_{n-1} + (\alpha_n - \alpha_{n-1})\rho\left(\frac{t - t_n}{s_n - t_n}\right),$$

$$u_n'' + a_n(t)u_n = 0, \qquad u_n(t_n) = u_{n-1}(t_n), \qquad u_n'(t_n) = u_{n-1}'(t_n) = 0.$$

Letting $\chi[t_n, t_{n+1})$ denote the characteristic function of the half-open interval, we set

$$a(t) = \sum_{n=1}^{\infty} a_n(t)\chi[t_n, t_{n+1}) \quad \text{and} \quad u(t) = \sum_{n=1}^{\infty} u_n(t)\chi[t_n, t_{n+1}).$$

It is clear that a(t) is a positive, nondecreasing function belonging to $C^\infty[0, \infty)$, and that u(t) satisfies the differential equation (1). Since $\alpha_n \to \infty$ as $n \to \infty$, it follows that $a(t) \to \infty$ as $t \to \infty$.

We now establish a bound on $a'(t)$. Differentiating a(t), we obtain the equation

$$a'(t) = \begin{cases} (\alpha_n - \alpha_{n-1})\rho'\left(\dfrac{t - t_n}{s_n - t_n}\right)\dfrac{1}{s_n - t_n} & \text{for } t_n \le t \le s_n, \\ 0 & \text{for } s_n \le t \le t_{n+1}. \end{cases}$$

Since $\rho(t)$ is a C^∞-function having compact support, $\rho'(t)$ is bounded by some positive number K. Hence, for $t_n \le t \le t_{n+1}$ and $n \ge 2$,

$$a'(t) \le K(\alpha_n - \alpha_{n-1})(s_n - t_n)^{-1} < 2K \log^{-m} n. \tag{3}$$

For $t_n \le t \le t_{n+1}$ and $n \ge 5$, it follows from the condition $n - 3/2 \le t_n \le n - 1$ that

$$\log n \ge \log t_{n+1} \ge \log t \ge 1.$$

Combining this with (3), we obtain the estimate $a'(t) = O(\log^{-m} t)$.

To show that $\limsup_{t\to\infty} |u(t)| > 0$, we choose numbers

$$\zeta_n = 2^{-1}(s_n - t_n)^2 a(s_n) \qquad (n \ge 2).$$

Since $\lim_{t\to\infty} t^{-1/2}\log^k t = 0$ for each positive integer k, there exists an integer N such that

$$16\pi^2 n^{-1/2}\log^{2m} n \le 1 \qquad \text{and} \qquad 2n^{-1/2}\log^{2m+1} n \le 1,$$

whenever $n \ge N$. Since each ζ_n $(n \ge N)$ satisfies the inequalities

$$\zeta_n = 2^{-1}\left[16\pi^2 + \sum_{k=1}^{n} k^{-1}\right] n^{-2}\log^{2m} n \le 2^{-1}[n^{-3/2} + (1 + \log n)n^{-2}\log^{2m} n]$$

$$\le 2^{-1}[n^{-3/2} + 2n^{-2}\log^{2m+1} n] \le n^{-3/2},$$

we see that $\sum_{n=1}^{\infty} \zeta_n < \infty$.

We now show that

$$|u(t_{n+1})| \ge [1 - \zeta_n]\,|u(t_n)| \tag{4}$$

for each of the points t_n. By Taylor's theorem,

$$u(s_n) = u(t_n) + (s_n - t_n)^2 u''(c)/2 \qquad (t_n \le c \le s_n).$$

We note that $|u''(c)| = a(c)\,|u(c)|$ and that $a(c) \le a(s_n)$. It is well known [5, Part 2, p. 28] that the values $|u(\xi_i)|$ determined by the points ξ_i $(i = 1, 2, \cdots)$ where $u'(\xi_i) = 0$ form a decreasing sequence. Therefore, $|u(c)| \le |u(t_n)|$. From these observations we obtain the relations

$$\begin{aligned} |u(s_n)| &= |u(t_n) + (s_n - t_n)^2 u''(c)/2| \\ &\ge [1 - 2^{-1}(s_n - t_n)^2 a(s_n)]\,|u(t_n)| = [1 - \zeta_n]\,|u(t_n)|. \end{aligned} \tag{5}$$

To estimate $|u(t_{n+1})|$, we integrate the expression $u'u'' + auu' = 0$ by parts and obtain the equation

$$a(t_{n+1})u^2(t_{n+1}) = (u'(s_n))^2 + a(s_n)u^2(s_n) + \int_{s_n}^{t_{n+1}} a'(t)u^2(t)\,dt.$$

From this we deduce that

$$u^2(t_{n+1}) \geq a(s_n)a^{-1}(t_{n+1})u^2(s_n) = u^2(s_n).$$

Combining (5) with this inequality, we obtain (4).

Since $\sum_{n=1}^{\infty} \zeta_n < \infty$, there exists a positive integer N such that $0 < \zeta_n < 1$ for $n \geq N$. From inequality (4), we see that

$$|u(t_{n+1})| \geq |u(t_N)| \prod_{k=N}^{n} (1 - \zeta_k).$$

Since the product $\prod_{k=N}^{\infty} (1 - \zeta_k)$ converges to some positive number, we deduce that $\limsup_{t \to \infty} |u(t)| > 0$, and this completes the proof.

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State University of New York at Buffalo
Amherst, New York 14226