ON A VARIATIONAL METHOD FOR UNIVALENT FUNCTIONS

Ch. Pommerenke

There are several proofs for the basic results on interior variations of univalent functions. The original proof of M. Schiffer [5] uses a variation of the Green's function and an approximation of the domain by smooth curves. The proof of G. M. Golusin [2], [3, p. 96] applies only to analytic variations (which are sufficient for almost all applications), and it uses the majorant method. For further discussions of this variational method, see [1], [4], [6].

We give a new proof of Golusin's version of the variational theorem. The proof is elementary, apart from the use of Carathéodory's kernel theorem. By univalent we mean analytic and univalent.

THEOREM. *Let* $f(z)$ *be univalent in* $|z| < 1$ *and normalized so that* $f(0) = 0$ *and* $f'(0) > 0$. *For* $0 < \lambda < \lambda_0$, *let* $g(z, \lambda)$ *be univalent in the annulus* $r < |z| < 1$, *where* r *is some fixed number. Let*

$$\frac{g(z, \lambda) - f(z)}{\lambda z f'(z)} \to \sum_{n=1}^{\infty} c_{-n} z^{-n} + c_0 + \sum_{n=1}^{\infty} c_n z^n \qquad (\lambda \to 0+), \tag{1}$$

locally uniformly in $r < |z| < 1$.

For $0 < \lambda < \lambda_0$, *let the univalent function* $f(z, \lambda)$ *map* $|z| < 1$ *onto the union of the doubly connected domain* $\{g(z, \lambda): r < |z| < 1\}$ *and the compact set enclosed by this domain, and let* $f(0, \lambda) = 0$ *and* $f'(0, \lambda) > 0$. *Then*

$$\frac{f(z, \lambda) - f(z)}{\lambda z f'(z)} \to \Re c_0 + \sum_{n=1}^{\infty} (c_n + \bar{c}_{-n}) z^n \qquad (\lambda \to 0+), \tag{2}$$

locally uniformly in $|z| < 1$.

Remark. The choice

$$g(z, \lambda) = f(z) + \frac{a\lambda f(z)^2}{f(z) - f(z_0)} \qquad (|z_0| < 1,\ |a| = 1,\ 0 < \lambda < \lambda_0)$$

leads to a special case of Schiffer's variational formula [5].

If S is the usual class of normalized univalent functions and $f(z)$ belongs to S, it follows from (2) that the function

$$f^*(z, \lambda) = f(z) + \left[(z f'(z) - f(z)) \Re c_0 + z f'(z) \sum_{n=1}^{\infty} (c_n + \bar{c}_{-n}) z^n \right] \lambda + o(\lambda)$$

belongs to S and is a variation of the function $f(z)$.

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Proof. (a) We may assume that $g(z, \lambda) \neq 0$ $(r < |z| < 1)$. We can write

$$g(z, \lambda) = f(\phi(z, \lambda), \lambda), \tag{3}$$

where $\phi(z, \lambda)$ is univalent in $r < |z| < 1$ and satisfies the condition $0 < |\phi(z, \lambda)| < 1$. Since, by (1), $g(z, \lambda)$ tends to $f(z)$ (as $\lambda \to 0$) locally uniformly in $r < |z| < 1$, Carathéodory's kernel theorem implies that $f(z, \lambda)$ tends to $f(z)$ (as $\lambda \to 0$) locally uniformly in $|z| < 1$. Hence $\phi(z, \lambda)$ tends to z (as $\lambda \to 0$) locally uniformly in $r < |z| < 1$. Since $|\phi(z, \lambda)| \to 1$ as $|z| \to 1-$, the reflection principle shows that $\phi(z, \lambda)$ is univalent in $r < |z| < 1/r$ and that $|\phi(z, \lambda)| = 1$ for $|z| = 1$.

(b) We define

$$p(z, \lambda) = \frac{f(z, \lambda) - f(z)}{\lambda z f'(z)}, \qquad q(z, \lambda) = \frac{g(z, \lambda) - f(z)}{\lambda z f'(z)}.$$

The function

$$\psi(z, \lambda) = \frac{1}{\lambda} \log \frac{\phi(z, \lambda)}{z} = \sum_{n=-\infty}^{\infty} b_n(\lambda) z^n \tag{4}$$

is analytic in $r < |z| < 1/r$, and $\Re\, \psi(z, \lambda) = 0$ for $|z| = 1$. It follows that $b_{-n}(\lambda) = -\overline{b_n(\lambda)}$ and $b_0(\lambda) = i\beta(\lambda)$ for some real $\beta(\lambda)$.

Equations (3) and (4) imply that

$$\psi(z, \lambda) = -\sum_{n=1}^{\infty} \overline{b_n(\lambda)} z^{-n} + i\beta(\lambda) + \sum_{n=1}^{\infty} b_n(\lambda) z^n = [q(z, \lambda) - p(z, \lambda)][1 + h(z, \lambda)], \tag{5}$$

where

$$h(z, \lambda) = \frac{\lambda \psi(z, \lambda)}{e^{\lambda \psi(z,\lambda)} - 1} \cdot \frac{(\phi(z, \lambda) - z) f'(z)}{f(\phi(z, \lambda), \lambda) - f(z, \lambda)} - 1.$$

Since $\lim_{\lambda \to 0+} \lambda \psi(z, \lambda) = 0$, locally uniformly in $r < |z| < 1$, the factor $\lambda \psi(z, \lambda)[e^{\lambda \psi(z,\lambda)} - 1]^{-1}$ tends to 1 as $\lambda \to 0$. Since $f(z, \lambda) \to f(z)$ and $\phi(z, \lambda) \to z$, the second factor also tends to 1. Hence

$$h(z, \lambda) \to 0 \qquad (\lambda \to 0+) \tag{6}$$

locally uniformly in $r < |z| < 1$.

(c) Choose ρ $(r < \rho^2 < \rho < 1)$, and set $M(\lambda) = \max_{|z| \leq \rho} |p(z, \lambda)| + 1$. By $\varepsilon_j(\lambda)$ $(j = 1, 2, 3, 4)$ we shall denote functions (independent of z) that tend to 0 as $\lambda \to 0+$. It follows from (5), (6), and (1) that

$$\left| \overline{b_n(\lambda)} + \frac{1}{2\pi i} \int_{|z| = \rho^2} q(z, \lambda) z^{n-1}\, dz - \frac{1}{2\pi i} \int_{|z| = \rho^2} p(z, \lambda) z^{n-1}\, dz \right| \leq \rho^{2n} M(\lambda) \varepsilon_1(\lambda)$$

for $n = 0, 1, \cdots$. The second integral vanishes for $n \geq 1$. Hence (1) shows that

$$|\overline{b_n(\lambda)} + c_{-n}| \leq \rho^{2n} M(\lambda) \varepsilon_2(\lambda) \qquad (n = 1, 2, \cdots).$$

The second integral is real for $n = 0$. Hence, by (1),

$$|-\beta(\lambda) + \Im c_0| \leq M(\lambda)\varepsilon_2(\lambda).$$

(d) For $|z| = \rho$, relations (5) imply the inequalities

$$\left|\psi(z, \lambda) - \sum_{n=1}^{\infty} c_{-n}z^{-n} - i\Im c_0 + \sum_{n=1}^{\infty} \bar{c}_{-n}z^n\right|$$

$$\leq \sum_{n=1}^{\infty} |\overline{b_n(\lambda)} + c_{-n}|\,\rho^{-n} + |\beta(\lambda) - \Im c_0| + \sum_{n=1}^{\infty} |b_n(\lambda) + \bar{c}_{-n}|\,\rho^n \tag{7}$$

$$\leq \sum_{n=0}^{\infty} (\rho^{-n} + \rho^n)\rho^{2n} M(\lambda)\varepsilon_2(\lambda) = M(\lambda)\varepsilon_3(\lambda).$$

Now relations (5), (1), and (6) imply that

$$M(\lambda) \leq K_1 + K_2 M(\lambda)\varepsilon_4(\lambda),$$

for some constant K_1. Therefore $M(\lambda) \leq K_3$. It follows from (5), (1), (6), and (7) that

$$p(z, \lambda) = q(z, \lambda) + [q(z, \lambda) - p(z, \lambda)]h(z, \lambda) - \psi(z, \lambda)$$

$$\to c_0 + \sum_{n=1}^{\infty} (c_n z^n + c_{-n}z^{-n}) - i\Im c_0 + \sum_{n=1}^{\infty} (\bar{c}_{-n}z^n - c_{-n}z^{-n}) \qquad (\lambda \to 0+),$$

and the last relation holds uniformly for $|z| = \rho$. This is equivalent to (2).

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Technische Universität Berlin