RESTRICTIONS OF ISOTOPIES AND CONCORDANCES

L. S. Husch and T. B. Rushing

If X and Y are polyhedra and h_0 and h_1 are piecewise-linear (PL) homeomorphisms of X onto Y, a *concordance (weak isotopy, pseudo-isotopy)* between h_0 and h_1 is a PL homeomorphism

$$H: X \times I \to Y \times I \qquad (I = [0, 1])$$

such that $H(x, 0) = (h_0(x), 0)$ and $H(x, 1) = (h_1(x), 1)$ for all $x \in X$. If, in addition, $H(x, t) = (h_t(x), t)$ for all $(x, t) \in X \times I$, then H is called an *isotopy*. If $A \subset X$, a concordance or isotopy H between h_0 and h_1 is *fixed on* A if $H(x, t) = (h_0(x), t)$ for all $(x, t) \in X \times I$.

Let M be a simply-connected PL manifold, and let m be an interior point of M. In [1], H. Gluck showed that if h_0 and h_1 are two isotopic (concordant) homeomorphisms of M, each of which leaves m fixed, then $h_0 \mid M - \{m\}$ and $h_1 \mid M - \{m\}$ are isotopic (concordant) homeomorphisms of $M - \{m\}$. We generalize as follows.

THEOREM. *Let Q^q be a PL q-dimensional manifold, and let M^m be a compact, connected, m-dimensional polyhedron with $q \geq m - 3$. Suppose that one of the following three properties is satisfied.*

a) Q *is* $(m + 1)$*-connected with* $q \geq 2m + 3$.

b) Q *is* $(m + 1)$*-connected, and* M *is a closed,* $(2m - q + 2)$*-connected* PL *manifold.*

c) Q *is* $(k + 1)$*-connected, and* M *is a* PL *manifold with* k*-spine* K^k $(k < n,\ q \geq m + k + 2)$.

Let $f: M^m \to \text{int}\, Q^q$ *be a* PL *embedding. If* h_0 *and* h_1 *are* PL *homeomorphisms of* Q *that are the identity on* $f(M^m)$ *and are isotopic (concordant), then there exists an isotopy (concordance) between* h_0 *and* h_1 *that is fixed on* $f(M^m)$.

Remark. This theorem can be generalized to consider the case where f is an allowable embedding. We shall assume familiarity with either [3] or [7].

1. CONCORDANCES

Let $H: Q \times I \to Q \times I$ be a concordance between h_0 and h_1. Define $F: M \times I \to Q \times I$ by $F(x, t) = (f(x), t)$. To prove the theorem in this case, it suffices to find a PL homeomorphism $h: Q \times I \to Q \times I$ such that $h \mid Q \times \{0, 1\}$ is the identity map and $hHF = F$, since hH is the desired concordance.

In case a), the existence of such an h is a well-known corollary of the general-position theorem. In case b), one applies Theorem 4 of [2]. The case where M is a bounded manifold is handled by techniques from the unpublished works of J. Dancis, J. F. P. Hudson, and R. Tindell. We sketch a proof, for the sake of completeness.

Since Q is $(k + 1)$-connected, there exists a PL homotopy $f_t: M \times I \to Q \times I$ such that $f_0 = F$ and $f_1 = HF$. We may assume that there exists an $\varepsilon > 0$ such that

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$f_t = f_0$ for $t \in [0, \varepsilon]$ and $f_t = f_1$ for $t \in [1 - \varepsilon, 1]$, and that

$$f_t \mid M \times ([0, \varepsilon] \cup [1 - \varepsilon, 1]) = f_0 .$$

Define f': $(M \times I) \times I \to (Q \times I) \times I$ by $f'(x, t) = (f_t(x), t)$. Move f' into general position, keeping

$$f' \mid (M \times I) \times ([0, \varepsilon] \cup [1 - \varepsilon, 1]) \cup M \times ([0, \varepsilon] \cup [1 - \varepsilon, 1]) \times I$$

fixed. Let S and B denote the singular set and branch set of f', respectively. Since

$$\dim B + \dim K \times I < m + 2,$$

we may assume that $B \cap K \times I \times I = \emptyset$ and that $f' \mid K \times I \times I$ is a PL embedding. One can easily find a regular neighborhood U of $K \times I \times I$ in $M \times I \times I$ such that

i) $f' \mid U$ is a PL embedding,

ii) there exists a $\delta > 0$ such that

$$M \times I \times ([0, \delta] \cup [1 - \delta, 1]) \cup M \times ([0, \delta] \cup [1 - \delta, 1]) \times I$$

is contained in U.

By the uniqueness theorem of regular neighborhoods, we may assume that f' is a PL embedding of $(M \times I) \times I$ into $(Q \times I) \times I$. Hence f_0 and f_1 are allowably concordant keeping $M \times \{0, 1\}$ fixed [2]; thus, by Corollary ii of [2], f_0 and f_1 are ambient isotopic keeping $Q \times \{0, 1\}$ fixed. Let ϕ_t be this ambient isotopy; $h = \phi_1$ is the desired PL homeomorphism.

2. Δ-SETS

We need the theory of Δ-sets due to C. P. Rourke and B. J. Sanderson [6] (or the quasi-simplicial sets of C. Morlet [5]). A Δ-set is essentially a semi-simplicial set without the degeneracies. We recall some of the basic definitions and results.

Let Δ^n denote the standard n-simplex with ordered vertices $v_0, v_1, \cdots, v_n$. The ith *face map* ∂_i: $\Delta^{n-1} \to \Delta^n$ is the order-preserving, simplicial embedding that omits v_i. Let Δ denote the category whose objects are Δ^n $(n = 0, 1, \cdots)$ and whose morphisms are generated by the face maps. A Δ-*set* is a contravariant functor from Δ to the category of sets. A Δ-*map* between Δ-sets is a natural transformation between the functors.

If X is a Δ-set, then $X^k = X(\Delta^k)$ is the set of *k-simplexes*, and the maps $\partial_i = X(\partial_i)$ are called *face maps*. We shall be interested in pointed Δ-sets in which we distinguish a simplex $*^k \in K^k$ for each k and designate $* \subset K$ as the sub-Δ-set of K consisting of these simplexes with $\partial_i *^k = *^{k-1}$.

With each ordered, simplicial complex K, we associate a Δ-set, also designated by K, whose k-simplexes are order-preserving, simplicial embeddings of Δ^k into K.

Let $\Lambda_{n,i} = \mathrm{Cl}(\mathrm{bdry}\, \Delta^n - \partial_i \Delta^{n-1})$. A Δ-set X is called a *Kan* Δ-*set* if every Δ-map f: $\Lambda_{n,i} \to X$ can be extended to a Δ-map f_1: $\Delta^n \to X$.

If X is a Kan Δ-set and P is a polyhedron, a *map* $f: P \to X$ is a Δ-map $f: K \to X$, where K is an ordered triangulation of P. The maps f_0 and f_1 ($f_i: P \to X$) are *homotopic* if there exists a map $F: P \times I \to X$ such that $F \mid P \times \{i\} = f_i$ $(i = 0, 1)$. $[P; X]$ denotes the set of homotopy classes. We need the following two propositions, which were proved by Rourke and Sanderson [6].

PROPOSITION 1. *Each homotopy class in* $[P; X]$ *is represented by a* Δ-*map* $f: K \to X$, *where* K *is some ordered triangulation of* P.

PROPOSITION 2. *Let* Q *be a subpolyhedron of* P, *and let*

$$h: Q \times I \cup P \times \{0\} \to X$$

be a Δ-*map to a Kan* Δ-*set* X; *then* h *extends to a* Δ-*map* $h': P \times I \to X$.

Let I^n denote the PL n-cell. If X is a pointed Kan Δ-set, then the nth *homotopy group* of X is given by the expression

$$\Pi_n X = [I^n, \operatorname{bdry} I^n; X, *],$$

where the quantity in brackets denotes the homotopy class of Δ-maps of pairs.

A Δ-map $\Pi: E \to B$ is called a *Kan fibration* if for all integers i and n, for each Δ-map $f: \Lambda_{n,i} \to E$, and for each extension $f_1: \Delta^n \to B$ of Πf, there exists an extension f' of f such that $f_1 = \Pi f'$.

PROPOSITION 3. *A Kan fibration of Kan* Δ-*sets has the homotopy lifting property for maps of polyhedra.*

3. SPACES OF PL EMBEDDINGS

Let $\operatorname{Aut}(Q)$ be the Δ-set whose r-simplexes are homeomorphisms $h: Q \times \Delta^r \to Q \times \Delta^r$ such that h is level-preserving (that is, $\rho h = \rho$, where ρ is the projection along the second factor). Define $\partial_i h = h \mid Q \times \partial_i \Delta^{r-1}$. Let $\operatorname{Aut}(Q \bmod f(M))$ be the sub-Δ-set of $\operatorname{Aut}(Q)$ consisting of r-simplexes h such that $h \mid f(M) \times \Delta^r$ is the identity map.

Let $PL(M, Q)$ be the Δ-set whose r-simplexes are PL embeddings $h: M \times \Delta^r \to Q \times \Delta^r$ such that h is level-preserving. Define $\partial_i h = h \mid M \times \partial_i \Delta^{r-1}$.

The proof of the following proposition is easy.

PROPOSITION 4. $\operatorname{Aut}(Q)$, $\operatorname{Aut}(Q \bmod f(M))$, *and* $PL(M, Q)$ *are Kan* Δ-*sets.*

Define $\Pi_f: \operatorname{Aut}(Q) \to PL(M, Q)$ by $\Pi_f(h) = hf_1$, where h is an r-simplex of $\operatorname{Aut}(Q)$ and $f_1(x, y) = (f(x), y)$ for $(x, y) \in M \times \Delta^r$.

PROPOSITION 5. $\Pi_f: \operatorname{Aut}(Q) \to PL(M, Q)$ *is a Kan fibration.*

Proof. Suppose we have the following commutative diagram of Δ-maps:

$$\begin{array}{ccc} \Lambda_{n,i} & \xrightarrow{g} & \operatorname{Aut}(Q) \\ \downarrow \cap & & \downarrow \Pi_f \\ \Delta^n & \xrightarrow{g_1} & PL(M, Q). \end{array}$$

It follows that g_1 can be represented as a level-preserving PL embedding $g_1 = M \times \Delta^n \to Q \times \Delta^m$.

By Remark 2 of [3, p. 154], there exists a level-preserving PL homeomorphism $G: Q \times \Delta^n \to Q \times \Delta^n$ such that $Gf_1 = g_1$. Hence G is an n-simplex of Aut(Q). Let

$$\tilde{G} = (G^{-1} \mid Q \times \Lambda_{n,i}) \circ g: Q \times \Lambda_{n,i} \to Q \times \Lambda_{n,i}.$$

Note that $\tilde{G}f \mid M \times \Lambda_{n,i} = G^{-1} gf \mid M \times \Lambda_{n,i} = G^{-1} g_1 \mid M \times \Lambda_{n,i} = f_1 \mid M \times \Lambda_{n,i}$. Hence $\tilde{G}: \Lambda_{n,i} \to \mathrm{Aut}(Q \bmod f(M))$ is a Δ-map that can be extended to

$$\tilde{G}_1: \Delta^n \to \mathrm{Aut}(Q \bmod f(M)).$$

Set $g_1' = G\tilde{G}_1: Q \times \Delta^n \to Q \times \Delta^n$. Then g_1' is the desired extension.

Let C(M, Q) be the Δ-set whose r-simplexes are continuous, level-preserving maps $g: M \times \Delta^r \to Q \times \Delta^r$. Choose the base-point Δ-set $*$ of C(M, Q) as follows. Set $*^r = f_1: M \times \Delta^r \to Q \times \Delta^r$, as defined above. The boundary operators are defined naturally.

PROPOSITION 6. *Under the conditions of the theorem,* $\Pi_1(C(M, Q))$ *is trivial.*

Proof. It suffices to show that if

$$g: M \times \mathrm{bdry}\, \Delta^2 \to Q \times \mathrm{bdry}\, \Delta^2$$

is a level-preserving map, then g can be extended to a level-preserving map $g': M \times \Delta^2 \to Q \times \Delta^2$. Let p be the projection $Q \times \mathrm{bdry}\, \Delta^2 \to Q$. Then pg can be extended to $g_0: M \times \Delta^2 \to Q$. Define $g'(x, y) = (g_0(x, y), y)$.

We can consider PL(M, Q) as a sub-Δ-set of C(M, Q).

PROPOSITION 7. *If* M *is a compact, connected,* m*-dimensional polyhedron and* Q *is a* PL, q*-dimensional manifold* $(q \geq 2m + i + 2)$, *then the homomorphism* $\Pi_i(PL(M, Q)) \to \Pi_i(C(M, Q))$ *induced by inclusion is an isomorphism.*

This proposition follows easily from the following two propositions, which are generalizations to product spaces of the simplicial-approximation and general-position theorems. The proofs are easy.

PROPOSITION 8. *Let* M *be a* PL *manifold without boundary, let* Y *be a* PL *manifold, and let* P, Q $(P \subseteq Q)$ *be compact polyhedra. Suppose* $f: Q \to M \times Y$ *is a continuous map such that* $f \mid P$ *is* PL. *There exists a homotopy*

$$h_t: M \times Y \to M \times Y \qquad (t \in I)$$

such that

i) $\rho_2 h_t = \rho_2$ *for* $t \in I$,

ii) $h_t f \mid P$ *is the identity for* $t \in I$, *and*

iii) $h_1 f: Q \to M \times Y$ *is* PL.

PROPOSITION 9. *Let* M *be a* PL *manifold without boundary, let* Y *be a* PL *manifold, and let* P, Q $(P \subseteq Q)$ *be compact polyhedra. Suppose* $f: Q \to M \times Y$ *is a* PL *map such that* $f \mid P$ *is a* PL *embedding. There exists a* PL *homotopy* $h_t: M \times Y \to M \times Y$ $(t \in I)$ *such that*

i) $\rho_2 h_t = \rho_2$ *for* $t \in I$,

ii) $h_t f \mid P$ *is the identity for* $t \in I$, *and*

iii) *the singular set of* $h_1 f$ *has dimension at most* $2 \dim Q - \dim(M \times Y)$.

4. ISOTOPIES

Let h_0 and h_1 be the isotopic PL homeomorphisms of Q that are the identity on $f(M^m)$. Let H be the isotopy. Then H represents a path in Aut(Q). Note that $\Pi_f(h_0) = \Pi_f(h_1)$; hence $\Pi_f H$ represents an element of $\Pi_1(PL(M, Q))$. By Propositions 6 and 7 and [4], $\Pi_1(PL(M, Q))$ is trivial. The path $\Pi_f G$ is homotopically equivalent to the base point $*^1 = f_1$ in PL(M, Q). By Proposition 3, this homotopy can be lifted to Aut(Q), which gives the desired isotopy between h_0 and h_1.

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The University of Georgia
Athens, Georgia 30601