ON THE TOPOLOGY OF A DUAL SPACE

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Let G and H be locally compact, connected, topological groups. Let Hom (G, H) denote the space of all continuous homomorphisms from G into H, with the compact-open topology (in other words, convergence on compacta is uniform). We shall call Hom (G, H) the *dual space of* G *with respect to* H.

In this note, we prove that the space Hom (G, H) is locally compact provided G and H are locally compact, connected, topological groups and H is finite-dimensional. As a corollary, we obtain the result that the automorphism group A(H) is locally compact in the compact-open topology (see [3]).

The proof of our main theorem consists of two parts. First, we prove that Hom (G, H) is locally compact if both G and H are finite-dimensional. Then we prove the local compactness of Hom (G, H) in the general case.

Throughout the paper, we assume that G and H are locally compact, connected, topological groups and that H is finite-dimensional. Let R be a compact subset of G, and let V be an open subset of H. We set

$$[R, V] = \{\sigma \in \text{Hom}(G, H) \colon \sigma(R) \subseteq V\},$$

and for $\rho \in$ Hom (G, H), we set

$$\langle \rho, R, V \rangle = \{\sigma \in \text{Hom}(G, H) \colon \rho(r)^{-1}\sigma(r) \in V \text{ for all } r \in R\}.$$

Then the collection

$$\{[R, V] \colon R \text{ is a compact subset of } G \text{ and } V \text{ is an open subset of } H\}$$

forms a basis for the topology for Hom (G, H). The collection

$$\{\langle \rho, R, V \rangle \colon \rho \in \text{Hom}(G, H),\ R \text{ is a compact subset of } G, \text{ and } V \text{ is a neighborhood of the identity of } H\}$$

also forms a basis of the topology of Hom (G, H). Since G is connected, we know that

$$\{\langle \rho, W, V \rangle \colon W \text{ is a fixed compact neighborhood of the identity of } G, \text{ and } V \text{ runs through the nuclei of } H\}$$

forms a basis of the topology of Hom (G, H) [1]. If A and B are subsets of G and H, respectively, then

$$\text{Hom}(G, A; H, B) = \{\sigma \in \text{Hom}(G, H) \colon \sigma(A) \subseteq B\}.$$

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We recall the following well-known structure theorem for locally compact groups.

STRUCTURE THEOREM. *Let* F *be a locally compact, connected group. Then we can find a neighborhood base of the identity* e, *composed of nuclei of the form* $W = K \times L$, *where* L *is a local Lie group and* K *is a compact subgroup. Moreover,* $[K, L] = e$.

The decomposition $W = K \times L$ is called the *Levi decomposition* of W [1].

Let $L^\sim = \bigcup_{n=1}^\infty L^n$. Then $L^\sim$ is a subgroup of F. We give $L^\sim$ the (unique) Lie-group topology, and we denote by L^* the Lie group so obtained. We then have the natural inclusion map i: $L^* \to L^\sim \subset F$. Let $D^\sim = K \cap L^\sim$. If $d^\sim \in K \cap L^\sim$ and $d^* = i^{-1}(d^\sim)$, then $F \approx \frac{K \times L^*}{D}$, where

$$D = \{(d, d^{*-1}): d \in K \cap L^\sim\}.$$

Since H is finite-dimensional, H has the Levi decomposition $V = K \times L$, where K is a totally disconnected, compact, central subgroup of H.

1. In this section, we assume that G is a finite-dimensional, locally compact, connected group. Let $\sigma \in \mathrm{Hom}(G, H)$. There is a neighborhood U of the identity e_1 of G such that $\sigma(U) \subseteq V = K \times L$ and $U = K_1 \times L_1$ (Levi decomposition). Therefore $\sigma(K_1) \subseteq K$ and $\sigma(L_1) \subseteq L$. This implies that

$$\sigma\left(\bigcup_{n=1}^\infty L_1^n\right) \subseteq \bigcup_{n=1}^\infty L^n.$$

Thus σ induces a homomorphism σ^* mapping L_1^* into L^* and D_1^* into D^*, where L_1^* is the (unique) Lie group obtained from $\bigcup_{n=1}^\infty L_1^n$, and where D_1^* is the discrete central subgroup of L_1^* corresponding to $D_1^\sim = K_1 \cap \left(\bigcup_{n=1}^\infty L_1^n\right)$. It is easy to see that σ^* is continuous, in other words, that

$$\sigma^* \in \mathrm{Hom}(L_1^*, D_1^*; L^*, D^*).$$

We note that for distinct homomorphisms $\sigma' \in \mathrm{Hom}(G, H)$, we might have to choose different neighborhoods $U' = K' \times L'$ such that $\sigma'(U') \subseteq V$. In each case, $L' \cap L_1$ is open in L_1 and in L'. Thus $\bigcup_{n=1}^\infty L^n = \bigcup_{n=1}^\infty L'^n$. Hence σ' defines a homomorphism σ'^* in $\mathrm{Hom}(L_1^*, L^*)$.

LEMMA 1.1. *There exists a continuous, one-to-one map*

$$\phi: \mathrm{Hom}(G, H) \to \mathrm{Hom}(L_1^*, L^*)$$

such that $\phi(\sigma) = \sigma^*$.

Proof. If $L_0 \subset L$ and $\langle \sigma^*; L_2, L_0 \rangle \subset \mathrm{Hom}(L_1^*, L^*)$, where L_2 is a compact neighborhood of the identity e_1 of L_1^*, and where L_0 is a neighborhood of the identity e of L^*, then there exists a compact subgroup K_2 of K_1 such that $K_2 \times L_2$ is a neighborhood of the identity of G and $\sigma(K_2 \times L_2) \subset K \times L_0$. Therefore

$$\phi(\langle \sigma; K_2 \times L_2, K \times L_0 \rangle) \subset \langle \sigma^*; L_2, L_0 \rangle,$$

and consequently ϕ is continuous.

If $\sigma \in \mathrm{Hom}(G, H)$ and $\phi(\sigma) = \sigma^* \in \mathrm{Hom}(L_1^*, D_1^*; L^*, D^*)$, then D_1^* is a finitely generated, discrete, central subgroup of L_1^* [1]. It is clear that

$$\mathrm{Hom}(L_1^*, D_1^*; L^*, D^*)$$

is a closed subgroup of $\mathrm{Hom}(L_1^*, L^*)$. Moreover, it is known that $\mathrm{Hom}(L_1^*, L^*)$ is locally compact [2]. Hence, $\mathrm{Hom}(L_1^*, D_1^*; L^*, D^*)$ is locally compact.

LEMMA 1.2. *The set* $A = \{\theta^* \in \mathrm{Hom}(L_1^*, D_1^*; L^*, D^*): \theta^* \mid D_1^* = \sigma^* \mid D_1^*\}$ *is an open subset of* $\mathrm{Hom}(L_1^*, D_1^*; L^*, D^*)$.

Proof. Since D_1^* is finitely generated, D_1^* has the generators $C_1, C_2, \cdots, C_n$. Since D^* is discrete, there exists a neighborhood W of the identity e of L^* such that $W \cap D = \{e\}$. Now it is easy to verify that

$$A = \mathrm{Hom}(L_1^*, D_1^*; L^*, D^*) \cap \langle \sigma^*; \{C_1, C_2, \cdots, C_n\}, W \rangle .$$

Thus A is an open subset of $\mathrm{Hom}(L_1^*, D_1^*; L^*, D^*)$.

LEMMA 1.3. *Let* $A^\sim = \{\sigma' \in \mathrm{Hom}(G, H): \sigma' \mid K_1 = \sigma \mid K_1\}$. *Then* $\phi \mid A^\sim$ *is a homeomorphism onto* A.

Proof. Because the relation $\theta^* \mid D_1^* = \sigma^* \mid D_1^*$ holds for each $\theta^* \in A$, the homomorphism θ^* induces a homomorphism from K_1 into K that agrees with σ on K_1. Thus θ^* induces a homomorphism $\tilde{\theta}$ such that

$$\tilde{\theta}: K_1 \times L_1^* \to K \times L^* \quad \text{and} \quad \theta: \frac{K_1 \times L_1^*}{D_1} \to \frac{K \times L^*}{D}.$$

This means $\phi(\theta) = \theta^*$ and $\phi(A^\sim) = A$. Suppose $\{\theta_n\}$ is a sequence in $A^\sim$. Since $\theta_n \mid K_1 = \sigma \mid K_1$, it follows that $\lim \theta_n = \theta \in A^\sim$ if and only if $\theta_n \mid L_1 \to \theta \mid L_1$ in the compact-open topology. This is equivalent to the condition $\lim_n \theta_n^* = \theta^*$. Hence ϕ is a homeomorphism.

PROPOSITION 1.4. $\mathrm{Hom}(G, H)$ *is locally compact.*

Proof. Since A is locally compact and open in $\mathrm{Hom}(L_1^*, L^*)$, the set $\phi^{-1}(A) = A^\sim$ is locally compact and open in $\mathrm{Hom}(G, K_1; H, V)$. By the definition of the topology on $\mathrm{Hom}(G, H)$, the set $\mathrm{Hom}(G, K_1; H, V)$ is open in $\mathrm{Hom}(G, H)$; therefore, $\mathrm{Hom}(G, H)$ is locally compact.

2. In this section, G is a locally compact, connected group. Let $\sigma \in \mathrm{Hom}(G, H)$. Let $U = K_1 \times L_1$ be a Levi decomposition of G such that $\sigma(U) \subseteq V$. Then $\sigma(K_1) \subseteq K$. Since K is totally disconnected, $\sigma(K_0) = e$, where K_0 denotes the identity component of K_1. Since K_1 is normal in G and K_0 is characteristic in K_1, it follows that K_0 is normal in G. Thus σ induces a homomorphism $\bar{\sigma}: G/K_0 \to H$.

LEMMA 2.1. $\mathrm{Hom}(G, K_1; H, K \times L)$ *is an open subset of* $\mathrm{Hom}(G, H)$, *and there exists a homeomorphism*

$$\psi: \mathrm{Hom}(G, K_1; H, K \times L) \to \mathrm{Hom}(G/K_0, K_1/K_0; H, K \times L).$$

Proof. The first part of the lemma follows from the definition of the topology. Define $\psi(\sigma) = \bar{\sigma}$. It is easy to verify that ψ is one-to-one and onto. If

$\sigma \in \mathrm{Hom}(G, K_1; H, K \times L)$, then σ is trivial on K_0. Thus the topology is determined by $(K_1/K_0) \times L_1$ and $K \times L$, and ψ is a homeomorphism.

THEOREM 2.2. *If* G *is a locally compact, connected, topological group and* H *is a locally compact, connected, finite-dimensional topological group, then the dual space* Hom (G, H) *is locally compact.*

Proof. Since G/K_0 is finite-dimensional, Proposition 1.4 implies that $\mathrm{Hom}(G/K_0, K_1/K_0; H, K \times L)$ is locally compact; thus $\mathrm{Hom}(G, K_1; H, K \times L)$ is a locally compact, open subset of Hom (G, H). Since $\sigma \in \mathrm{Hom}(G, H)$, it follows that $\sigma \in \mathrm{Hom}(G, K'; H, K \times L)$ for some compact K'. Hence, Hom (G, H) is locally compact.

If G = H, then Hom (H, H) forms a topological semigroup, if we define $\sigma_1 \sigma_2$ by composition [1]. Let

$$A(H) = \{\sigma \in \mathrm{Hom}(H, H): \sigma \text{ is one-to-one and onto}\}.$$

Then $\sigma^{-1} \in A(H)$. Also, A(H) with the relative topology forms a topological group. Let $S = \overline{A(H)} \subseteq \mathrm{Hom}(H, H)$. Then S is a locally compact semigroup with a dense topological subgroup A(H). Hence A(H) is an open subset of S [4], and it is locally compact. We have now proved the following theorem.

THEOREM 2.3 (see [3]). *Let* H *be a locally compact, connected, finite-dimensional topological group, and let* A(H) *be the group of automorphisms of* H. *Then* A(H) *is locally compact in the compact-open topology.*

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