THE BROUWER PROPERTY AND INVERT SETS

S. F. Kapoor

1. INTRODUCTION

A topological space X is said to have the *Brouwer property* if homeomorphic images of open subsets of X are also open subsets of X (see G. T. Whyburn [9], [10], and [11]). Thus, euclidean spaces and manifolds have the Brouwer property, whereas manifolds with nonempty boundary do not. For $n < 3$, E. Duda [3] showed that an n-complex has the Brouwer property if and only if it is an n-manifold.

Invertible spaces were introduced by P. H. Doyle and J. G. Hocking [2]; a point p of a topological space X is an *invert point* if for each open neighborhood U of p there exists a homeomorphism h of X onto itself such that $h(X - U) \subseteq U$. If h is isotopic to id_X, then p is a *continuous invert point*. The collection of all invert points is the *invert set*, denoted by I(X). The *continuous invert set* CI(X) is defined similarly. Doyle [1] investigated invert sets in complexes, and he showed that for each complex K, the set I(K) is the empty set, a point, or a simplicial sphere. Hocking proved that if $I(K) = S^k$ $(0 \leq k \leq n)$, then the n-complex K is a multiple suspension. An n-complex K with a single-point invert set was characterized by Doyle [1] and by V. M. Klassen [7] for n = 1 and 2. In this paper, we discuss n-complexes having the Brouwer property, and we focus our attention on the case where n = 3 and I(K) is a single point.

2. A CHARACTERIZATION OF THE 3-SPHERE

It is easily seen that if K is an n-complex with the Brouwer property and $I(K) = \{p\}$, then Lk(p) has the Brouwer property. Also, a complex L has the Brouwer property if its suspension $\mathscr{S}(L)$ has the Brouwer property.

THEOREM 1. *Let* K *be a* 3-*complex with the Brouwer property. Then* $\dim \{I(K)\} \geq 1$ *if and only if* $K = S^3$.

Proof. If $K = S^3$, then $I(K) = S^3$. On the other hand, if $\dim\{I(K)\} \geq 1$, we can write $K = \mathscr{S}(L)$, where L is a 2-complex with the Brouwer property. By Duda's result, L is a 2-manifold. Moreover, there exist x and y in L such that $\{x, y\} \subseteq L \cap I(K)$. But since L is a manifold, $L \subseteq I(K)$. Thus $K = \mathscr{S}(L) \subseteq I(K)$. Consequently, $K = I(K)$, and by [2], $K = S^3$.

3. ORBITS

Let K be a 3-complex, with $I(K) = S^0$, and possessing the Brouwer property. Then $K = \mathscr{S}(L)$, where L is a 2-manifold M^2. It is possible that M^2 is a disjoint union of m 2-manifolds $(m \geq 1)$. From such a complex we can obtain another with a single-point invert set, by identifying the two suspension points of $\mathscr{S}(L)$ (see Theorem 3).

Received August 17, 1968.
This research forms part of a doctoral thesis written under the direction of Professor P. H. Doyle at Michigan State University in 1967.

First we note that if $n^*(K)$ denotes the number of isotopy orbits of an n-complex K and $p \in I(K)$, then (i) $I(K) = S^0$ implies that $n^*(K) = 0$, (ii) $n^*(K) = 1$ if and only if $K = S^n$ for $n \geq 1$ or $K = \{p\}$, and (iii) $n^*(K) = 2$ and $I(K) = \{p\}$ imply that $K - \{p\}$ is locally euclidean of dimension n. We remark that $n^*(K) = 2$ does not imply $n^*(\mathscr{S}(K)) = 2$; however the inequality $1 \leq n^*(\mathscr{S}(K)) \leq 4$ always holds.

THEOREM 2. *Let* K *be a connected* n-*complex with* $p \notin I(K)$. *If* $\dim\{I(K)\} = k$ *and* d *denotes the dimension of the isotopy orbit of* p, *then* $d > k$.

Proof. The proof is by induction on k. When $k = -1$, then $I(K) = \emptyset$ and $d \geq 0$. For $k = 0$, $I(K)$ is a point or a 0-sphere. But $p \notin I(K)$ implies that p is not a singularity of K and that $d \geq 1$. Assume that the result is true for all $k < m$. Let K be a connected n-complex with $\dim\{I(K)\} = m \geq 1$. Let $p \notin I(K)$ and $d \leq m$. Under some triangulation T of K, let the isotopy orbit of p be written as a union of open simplices, and let L be the closure of this orbit. Then L is a subcomplex of K under T, and $\dim L \leq m$. Now $I(K) = S^m$, and each simplex of $I(K)$ is principal. Also, $S^m \cap L \neq \emptyset$. Let $M = S^m \cup L$ be a subcomplex of K under T. Then $\dim M = m$ and $S^m \subseteq I(M)$. This implies that $M = S^m$ and $L = \emptyset$. This is a contradiction. Hence $d > m$.

THEOREM 3. *Let* K *be a* 3-*complex with the Brouwer property and with* $I(K) = \{p\}$. *If* $n^*(K) = 2$, *then* K *is a suspension of a closed* 2-*manifold with the suspension points identified at* p.

Proof. Recall that Lk(p) has the Brouwer property. Since $\dim\{Lk(p)\} = 2$, Lk(p) is a closed 2-manifold M^2. Out of the two orbits under isotopy, one orbit is required for p. This shows that K contains no simplex of dimension less than or equal to $(i - 1)$ that is not a face of an i-simplex in K for $0 \leq i \leq 3$. Moreover, Lk(p) must have precisely two components, for if it has one, then $I(K) \supset \{p\}$ ($\supset$ and $\subset$ denote strict inclusion). The result now follows.

THEOREM 4. *Let* K *be a* 3-*complex with the Brouwer property and with* $I(K) = \{p\}$. *Suppose that* $n^*(K) = 3$. *Then either*

(i) $K = K_1 \cup K_2$, *where* $K_1 \cap K_2 = \{p\}$, *and for* $i = 1, 2$, *the complex* K_i *is a suspension of a* 2-*manifold with the suspension points identified at* p *or a cone over a* 2-*manifold from* p, *or*

(ii) K *is a suspension over a* 2-*manifold with the suspension points identified at* p.

Proof. The proof proceeds as in the last theorem. However, since $n^*(K) = 3$, it may happen that K is the union of two 3-complexes K_1 and K_2, with $K_1 \cap K_2 = \{p\}$ and each K_i behaving as in Theorem 3. This gives the first part of (i). It may also happen that $Lk(p) \cap K_i$ is connected, in which case we get a cone over a 2-manifold from p. This completes the proof of (i), and (ii) follows by arguments similar to those used earlier.

Let K be an n-complex with $I(K) = \{p\}$. Suppose $x \in K - \{p\}$ and $\dim(\mathfrak{U}) = k$ is minimal, where $\mathfrak{U}$ is the isotopy orbit of x. Then $\overline{\mathfrak{U}} = \mathfrak{U} \cup \{p\}$ and $p \in Cl(\overline{\mathfrak{U}})$. Also, $\overline{\mathfrak{U}} - \{p\}$ is a k-manifold M^k with $\partial M^k = \emptyset$. By an earlier remark, M^k has the Brouwer property, and consequently the same is true of $Lk(p, \overline{\mathfrak{U}})$. If $k = 1$, then $\overline{\mathfrak{U}} = S^1$. If $k = 2$, then $Lk(p, \overline{\mathfrak{U}})$ has dimension 1 and possesses the Brouwer property, and therefore it is a collection of disjoint 1-spheres. If $Lk(p, \overline{\mathfrak{U}})$ is a 1-sphere, then $\overline{\mathfrak{U}} = S^2$. If $Lk(p, \overline{\mathfrak{U}})$ is a collection of two disjoint 1-spheres, then $\overline{\mathfrak{U}}$ is a pinched torus. If $k = 3$, then $Lk(p, \overline{\mathfrak{U}})$ has dimension 2 and possesses the Brouwer property, and it must be a 2-manifold without boundary. This proves the next result.

THEOREM 5. *Let* K *be an* n-*complex with* $I(K) = \{p\}$. *Suppose* $x \in K - \{p\}$ *and* $\dim(\mathfrak{U}) = k$ *is minimal, where* $\mathfrak{U}$ *is the isotopy orbit of* x. *Then*

(i) $k = 1$ *implies that* $\overline{\mathfrak{U}} = S^1$,

(ii) $k = 2$ *implies that* $Lk(p, \overline{\mathfrak{U}})$ *is a collection of disjoint* 1-*spheres, and*

(iii) $k = 3$ *implies that* $Lk(p, \overline{\mathfrak{U}})$ *is a* 2-*manifold without boundary.*

In particular, the preceding result is useful for a 3-complex, where the possible values of k are precisely 1, 2, and 3.

4. SINGLE-POINT INVERT SETS

It was conjectured in [5] that if K is any complex with a single-point invert set, then $I(\mathscr{S}(K))$ must be a 0-sphere. In this section we shall discuss partial results in this direction, using the Brouwer property and isotopy orbits. It is known (see [6]) that if $I(K) = \{p\}$ and $I(\mathscr{S}(K)) \supset S^0$, then $I(\mathscr{S}(K)) \supseteq S^2$. Also, if $I(K) = \{p\}$, then $p \in I(\mathscr{S}(K))$ if and only if $\dim\{I(\mathscr{S}(K))\} \geq 1$.

THEOREM 6. *Let* K *be a* 1-*complex with* $I(K) = \{p\}$. *Then* $I(\mathscr{S}(K)) = S^0$.

Proof. Let $q \in K - \{p\}$, and let U be an open neighborhood of q in $\mathscr{S}(K)$. We can take U to be an open 2-cell. Clearly, there exists no homeomorphism h of $\mathscr{S}(K)$ onto itself such that $h(\mathscr{S}(K) - U) \subseteq U$. In particular, h(p) cannot lie in U. Hence $q \in K - \{p\}$ implies that $q \notin I(\mathscr{S}(K))$ and therefore $|K \cap I(\mathscr{S}(K))| \leq 1$. If $I(\mathscr{S}(K)) \supset S^0$, then $\dim\{I(\mathscr{S}(K))\} \geq 2$ and $|K \cap I(\mathscr{S}(K))| \geq 2$. Thus $K \cap I(\mathscr{S}(K)) = \emptyset$, and the result follows.

THEOREM 7. *Let* K *be an* n-*complex with* $I(K) = \{p\}$ *and* $n^*(K) = 2$. *If* $\dim\{I(\mathscr{S}(K))\} \geq 1$, *then* $\mathscr{S}(K) = S^{n+1}$.

Proof. Since $n^*(K) = 2$, the set $K - \{p\}$ is locally euclidean of dimension n. Also, $\dim\{I(\mathscr{S}(K))\} \geq 1$ implies that some $x \in K - \{p\}$ lies in $I(\mathscr{S}(K))$. By homogeneity, $K - \{p\} \subseteq I(\mathscr{S}(K))$; therefore $K \subseteq I(\mathscr{S}(K))$, since $p \in I(\mathscr{S}(K))$. Now $\dim K = n$ implies that $\dim\{I(\mathscr{S}(K))\} = \dim\{\mathscr{S}(K)\} = n + 1$. This completes the proof.

COROLLARY 8. *Let* K *be an* n-*complex with* $I(K) = \{p\}$ *and* $n^*(K) = 2$. *If* $p \in I(\mathscr{S}(K))$, *then* K *and* Lk(p, K) *have the Brouwer property.*

Proof. By an earlier remark, $p \in I(\mathscr{S}(K))$ implies that $\dim\{I(\mathscr{S}(K))\} \geq 1$. The preceding theorem implies that $\mathscr{S}(K) = S^{n+1}$, and S^{n+1} has the Brouwer property. By another remark, both K and Lk(p, K) have the Brouwer property.

For an n-complex K with $I(K) \neq \emptyset$, assume that $p \in I(K)$ and St(p) embeds in E^n. Now suppose that $\mathscr{S}(K)$ has the Brouwer property and $\dim\{I(\mathscr{S}(K))\} \geq 1$. Then K has the Brouwer property, $K = S^n$, and $\mathscr{S}(K) = S^{n+1}$. Consider the case where $I(K) = \{p\}$. If $\mathscr{S}(K)$ has the Brouwer property, then $\dim\{I(\mathscr{S}(K))\} < 1$, or $I(\mathscr{S}(K)) = S^0$. This leads to the following.

THEOREM 9. *Let* K *be an* n-*complex* $(n \geq 1)$ *such that* $p \in I(K)$ *and* St(p) *embeds in* E^n. *Moreover, let* $\mathscr{S}(K)$ *have the Brouwer property. Then*

(i) $\dim\{I(\mathscr{S}(K))\} \geq 1$ *implies that* $K = S^n$,

(ii) $I(K) = \emptyset$ *or* $I(k) = \{p\}$ *implies that* $I(\mathscr{S}(K)) = S^0$, *and*

(iii) $I(K) \neq S^0$.

Proof. We need only show (iii). Assume the contrary. Then $K = \mathscr{S}(L)$ and $\mathscr{S}(K) = \mathscr{S}^2(L)$. By Theorem 7 of [1], $\dim\{I(\mathscr{S}(K))\} \geq 1$. Using (i), we see that $K = S^n$, and since $n \geq 1$, this contradicts the assumption that $I(K) = S^0$.

THEOREM 10. *Let* K *be a* 2-*complex with* $I(K) = \{p\}$ *and* $n^*(K) = 2$. *Then* $I(\mathscr{S}(K)) = S^0$.

Proof. Assume that $\dim\{I(\mathscr{S}(K))\} \geq 1$. By Theorem 7, $\mathscr{S}(K) = S^3$. This contradicts Klassen's characterization (see [7]) of a 2-complex with a single-point invert set.

Suppose $n > 1$, and identify at p two antipodal points of S^n in a nice way to obtain an n-complex K. This may be called a generalized pinched torus. It is evident that $I(K) = \{p\}$ and $n^*(K) = 2$. Moreover, $I(\mathscr{S}(K)) = S^0$, since $\mathscr{S}(K) \neq S^{n+1}$. This suggests that if K is an n-complex with $I(K) = \{p\}$ and $n^*(K) = 2$, and if K is not a homotopy n-sphere, then $I(\mathscr{S}(K)) = S^0$.

We can prove this assertion by using Theorem 7. If K is an n-complex $(n \geq 2)$ such that $\mathscr{S}(K) = S^{n+1}$, that is, if K is a homotopy n-sphere, let v be any vertex of K in a given triangulation. Then K and Lk(v, K) have the Brouwer property. Since $\text{Lk}(v, \mathscr{S}(K)) = \mathscr{S}(\text{Lk}(v, K))$ and $\mathscr{S}(K) = S^{n+1}$, $\mathscr{S}(\text{Lk}(v, K))$ has the integral homology groups of an n-sphere. Moreover, for $2 \leq i \leq n$,

$$\Phi: H_{i-1}(\text{Lk}(v, K)) \to H_i(\mathscr{S}(\text{Lk}(v, K)))$$

is an onto isomorphism with $H_0(\text{Lk}(v, K)) = Z$. Thus

$$H_k(\text{Lk}(v, K)) = \begin{cases} 0 & \text{for } 1 \leq k \leq n-2, \\ Z & \text{for } k = 0,\ n-1. \end{cases}$$

Remark. We can show that the local homology groups are invariant under all triangulations of K, by using the uniqueness of the open-cone neighborhood (see [8]). Let v be any vertex of K under any triangulation. Consider Int(St(v)) - v. There exists a deformation of this onto Lk(v). Now Int(St(v)) is an open-cone neighborhood of v. By Kwun's theorem, we deduce that the links of v are homeomorphic under all triangulations of K. This proves the assertion.

THEOREM 11. *Let* K *be an* n-*complex with* $n \geq 2$, $I(K) = \{p\}$, *and* $n^*(K) = 2$. *Let* v *be any vertex of* K *under the given triangulation, and suppose that either*

(i) $H_k(\text{Lk}(v, K)) \neq 0$ *for some* k $(1 \leq k \leq n-2)$, *or*

(ii) $H_k(\text{Lk}(v, K)) \neq Z$ *for* $k = 0$ *or* $k = n-1$.

Then $I(\mathscr{S}(K)) = S^0$.

Proof. By Theorem 7, the denial of the assertion produces a contradiction to the preceding remarks.

THEOREM 12. *Let* K *be a* 3-*complex with* $I(K) = \{p\}$ *and* $n^*(K) = 2$. *Then* $I(\mathscr{S}(K)) = S^0$.

Proof. Assume that $\dim\{I(\mathscr{S}(K))\} \geq 1$. By Theorem 7, $\mathscr{S}(K) = S^4$. Also, earlier remarks imply that

$$H_0(\text{Lk}(p, K)) = H_2(\text{Lk}(p, K)) = Z \quad \text{and} \quad H_1(\text{Lk}(p, K)) = 0.$$

Moreover, Lk(p, K) has the Brouwer property, by Corollary 8. By Duda's result (see [3]), Lk(p, K) is a 2-manifold without boundary with the prescribed homology groups. Thus Lk(p, K) = S^2 and the set K = p · Lk(p, K) is a 3-cell with a 2-sphere of invert points. This contradicts the hypothesis that I(K) = $\{p\}$.

As we mentioned earlier, these results are special verifications of the conjecture that the 0-sphere is the invert set of the suspension of any complex possessing a single-point invert set. It would be desirable to drop the restriction on the number of isotopy orbits, or even to obtain a characterization of a 3-complex with a single-point invert set.

REFERENCES

1. P. H. Doyle, *Symmetry in geometric complexes.* Amer. Math. Monthly 73 (1966), 625-628.

2. P. H. Doyle and J. G. Hocking, *Invertible spaces.* Amer. Math. Monthly 68 (1961), 959-965.

3. E. Duda, *Brouwer property spaces.* Duke Math. J. 30 (1963), 647-660.

4. J. G. Hocking, *Invert sets in polyhedra.* Amer. Math. Monthly 75 (1968), 357-362.

5. S. F. Kapoor, *Inverting and monotone properties of complexes.* Ph.D. Thesis, Michigan State University, 1967.

6. ———, *Invert sets in complexes and their suspensions* (to appear).

7. V. M. Klassen, *Complexes with invert points.* Ph.D. Thesis, Virginia Polytech. Inst., 1964.

8. K. W. Kwun, *Uniqueness of the open cone neighborhood.* Proc. Amer. Math. Soc. 15 (1964), 476-479.

9. G. T. Whyburn, *Topological analysis.* Bull. Amer. Math. Soc. 62 (1956), 204-218.

10. ———, *On the invariance of openness.* Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 464-466.

11. ———, *Compactness of certain mappings.* Amer. J. Math. 81 (1959), 306-314.

Western Michigan University
Kalamazoo, Michigan 49001