A NONSTANDARD APPROACH TO LINEAR FUNCTIONS

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It is a well-known result, due to Cauchy, that all of the continuous solutions of the functional equation

(1) $$f(x + y) = f(x) + f(y),$$

where f is required to be a real-valued function of a real variable, are given by $f(x) = mx$. In 1905, Hamel discovered discontinuous solutions of (1). His construction depended on the existence of a basis for the vector space R of real numbers over the rational field [2]. Further papers on the subject have dealt mainly with the problem of finding conditions on an additive function that guarantee its continuity. For example, one such condition, due to G. Darboux, is that the function be bounded on some interval [1, p. 109]. A nonstandard discussion of this condition may be found in [4, pp. 80-83].

The dichotomy between the continuous and discontinuous solutions of (1) is striking, especially in view of the simplicity of the equation. In the present paper, we use the methods of nonstandard analysis to investigate the question whether all solutions of (1) can be described in terms of linear functions of the form mx, where m may be either infinite or finite.

The problem of finding solutions to (1) is closely related to the problem of finding all the characters of an additive subgroup S of R, that is, all the complex-valued functions χ on S such that $|\chi(x)| = 1$ for all $x \in S$ and

(2) $$\chi(x + y) = \chi(x)\chi(y)$$

for all $x, y \in S$.

The solutions of this character problem are of two kinds, continuous and discontinuous, all of the former having the form $\chi(x) = e^{imx}$ ($m \in R$). The following version of Kronecker's approximation theorem (see [3, p. 431]) gives important information about the discontinuous solutions of (2).

THEOREM. *If* S *is a subgroup of* R *and* χ *is a character of* S, *then for every* $\varepsilon > 0$ *and every finite subset* $\{x_1, \cdots, x_n\}$ *of* S, *there exists a continuous character* $\chi_0(x) = e^{imx}$ *such that*

$$|\chi(x_j) - \chi_0(x_j)| < \varepsilon \quad (j = 1, 2, \cdots, n).$$

It follows from the approximation theorem that for each character χ of a subgroup S of R, the relation $T_1(x, \varepsilon, m)$ defined by the inequality $|\chi(x) - e^{imx}| < \varepsilon$ is concurrent in the sense of A. Robinson [5, p. 31]. Hence there exists an element m of an enlargement *R of R such that $\chi(x) \simeq e^{imx}$ for all $x \in S$. Thus every character χ of S, continuous or not, is of the form $\chi(x) = {}^0(e^{imx})$ for some $m \in {}^*R$.

Received September 5, 1968.
This work was supported by National Science Foundation Grant No. GP-6955.

To deal with additive functions, we need another form of the approximation theorem.

THEOREM. *Suppose* f *is a real-valued function on a subgroup* S *of* R *such that*

$$f(x+y) \equiv f(x) + f(y) \pmod{N} \tag{3}$$

for some positive number N. *Then, for each* $\varepsilon > 0$ *and every finite subset* $\{x_1, x_2, \cdots, x_n\}$ *of* S, *there exists a number* m *such that*

$$|f(x_j) - mx_j| < \varepsilon \pmod{N} \quad (j = 1, \cdots, n),$$

in other words, such that $|f(x_j) - mx_j|$ *is always within* ε *of some integral multiple of* N.

Proof. Each character on S is of the form $\chi(x) = \exp(2\pi i f(x)/N)$, where f satisfies (3). By the character approximation theorem, there exists for each $\delta > 0$ a number m such that $|\chi(x_j) - \exp(2\pi i m x_j/N)| < \delta$ $(j = 1, \cdots, n)$. Since the logarithmic function is continuous, it follows that to each $\varepsilon > 0$ there corresponds a number m such that

$$|2\pi f(x_j)/N - 2\pi m x_j/N| < 2\pi\varepsilon/N \pmod{2\pi} \quad (j = 1, \cdots, n),$$

in other words, such that $|f(x_j) - mx_j| < \varepsilon \pmod{N}$ $(j = 1, \cdots, n)$.

For infinite values of m, the linear function mx is clearly infinite for all nonzero standard x. Thus, to obtain a standard additive function, we must reduce the values of the function mx to finite values. Let K be the additive subgroup of *R generated by the set of infinite powers of 2; that is, let

$$K = \{z2^N \mid z \text{ and } N \text{ integers, } N > 0 \text{ and infinite}\}.$$

If K_n is the (additive) subgroup of *R generated by 2^n (n a standard positive integer), then K can also be written $K = \bigcap K_n$ $(n = 1, 2, \cdots)$.

THEOREM. *For* $m \in {}^*R$, *let* S *denote the subgroup of* R *consisting of all* x *such that* mx *is finite modulo* K, *in other words, such that* mx *is congruent to some finite number* z(x) *modulo* K. *Define a function* f *on* S *by* $f(x) = {}^0(z(x))$. *Then* f *is additive on* S. *Conversely, if* f *is additive on a subgroup* S *of* R, *then there exists a number* m *in* *R *such that for each* $x \in S$, mx *is finite modulo* K *and* $f(x) \cong mx \pmod{K}$. ($a \cong b \pmod{K}$ means that the distance between a - b and some element of K is infinitesimal.)

Proof. Suppose $m \in {}^*R$. If $x, y \in R$ have the property that mx and my are finite modulo K, then certainly $m(x+y) = mx + my$ has the same property. Now, by definition of f,

$$f(x) + f(y) \cong mx + my = m(x+y) \cong f(x+y) \quad \text{modulo } K.$$

Since the values of f are standard and no two standard numbers can be congruent modulo K without being equal, $f(x) + f(y) = f(x+y)$.

Now suppose f is an additive function on a subgroup S of R. Then certainly

$$f(x+y) \equiv f(x) + f(y) \pmod{2^n}$$

for every positive integer n. By the second approximation theorem, for any $\varepsilon > 0$, for positive integers $n_1, \cdots, n_p$, and for numbers $x_1, \cdots, x_k \in S$, there exists some $m \in R$ such that

$$|f(x_j) - mx_j| < \varepsilon \pmod{2^i} \quad (j = 1, \cdots, k;\ i = n_1, \cdots, n_p),$$

for we can take N to be the largest of the numbers 2^i $(i = n_1, \cdots, n_p)$. The relation $T_2(\varepsilon, x, n, m)$ defined by the inequality $|f(x) - mx| < \varepsilon \pmod{2^n}$ is therefore concurrent; hence, there exists an m in $^*R$ such that, for every $x \in S$ and every finite positive integer i,

$$f(x) \cong mx \pmod{2^i}.$$

This conclusion implies that $f(x) \cong mx \pmod{K}$, and the proof is complete.

Since the elements of K are fairly numerous, in the sense that each interval of infinite length in $^*R$ contains infinitely many elements of K, it is natural to ask whether, for each infinite $m \in {}^*R$, mx is necessarily finite modulo K for every $x \in R$. The following theorem answers this question in the negative.

THEOREM. *There exists a point* $y \in {}^*R$ *such that the distance from* y *to each element of* K *is infinite.*

Proof. Consider the sequence of numbers defined inductively by the conditions $y_1 = 1$, $y_{n+1} = 2^n - y_n$. If $\|x\|_n$ denotes the distance from x to the nearest integral multiple of 2^n, then a straightforward induction argument shows that

$$\|y_n\|_i = y_i \quad (i = 1, 2, \cdots, n)$$

for each n. Hence the relation $T_3(n, y)$ defined by the statement $\|y\|_n = y_n$ is concurrent; therefore there exists a $y \in {}^*R$ such that $\|y\|_n = y_n$ for every standard positive integer n. Now suppose y is within a finite distance, say 2^k, of some element of K. This element is evidently a multiple of 2^{k+j} for each standard j, and since the sequence $\{y_i\}$ $(i = 1, 2, \cdots)$ increases without bound, eventually we would have the contradictory inequalities

$$\|y\|_{k+j} < 2^k < y_{k+j}.$$

Thus y is infinitely distant from each element of K.

The following problem remains open: to characterize the group of elements $m \in {}^*R$ such that mx is finite modulo K for all $x \in R$.

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