WEAKLY FLAT SPHERES

Paul F. Duvall, Jr.

1. INTRODUCTION

In [7], D. R. McMillan used the cellularity criterion to give a sufficient condition for the complementary domains of a topologically embedded (n - 1)-sphere in the n-sphere S^n to be open n-cells. In general, if $\Sigma^k \subset S^n$ is a topologically embedded k-sphere, one may ask for conditions that guarantee that the complement $S^n - \Sigma^k$ is homeomorphic to the complement of the standard k-sphere in S^n. In other words, when is $S^n - \Sigma^k$ homeomorphic to $S^{n-k-1} \times R^{k+1}$? When this homeomorphism occurs, we follow Rosen [9] and say that Σ^k is *weakly flat*.

For $k \geq 0$, let D^k be the standard k-cell in Euclidean space R^k. If X is a space, a *loop* in X is a continuous function from ∂D^2 into X. The loop f: $\partial D^2 \to X$ is *null homotopic* if f has a continuous extension F: $D^2 \to X$. In this paper, we study weak flatness *via* the following generalization of the cellularity criterion.

Definition. Let X be a closed set in the interior of a manifold M. We say that M - X *is* 1 - ℓc *at* X if each open neighborhood U of X in M contains an open neighborhood V of X such that each loop in V - X is null homotopic in U - X.

In Section 2, we give an argument similar to that of L. C. Siebenmann in [10] to show that, for $n \geq 5$ and $2 \leq k \leq n - 3$, $\Sigma^k \subset S^n$ is weakly flat if and only if $S^n - \Sigma^k$ is 1 - ℓc at Σ^k. Section 3 is devoted to a proof that under certain conditions, if X is a compact ANR in S^n, if S^n - X is 1 - ℓc at X, and if Y is obtained from X by the deletion of open cones, then S^n - Y is 1 - ℓc at Y. In Section 4 we apply the results in Sections 2 and 3 to questions about weak flatness and cellularity. For example, we show that with dimensional restrictions the boundary of a cellular k-cell in S^n is a weakly flat sphere and that weak flatness is in a certain sense transitive (Theorem 4.1). Finally, in Section 5 we give an example to show that a weakly flat sphere need not be locally flat at any point.

Often we shall indicate the dimension of a space by a superscript the first time it appears in the discussion, and omit the superscript thereafter. We abbreviate *piecewise linear* to PL, throughout. "X ≈ Y" is to be read as "X *is homeomorphic to* Y" if X and Y are spaces, and as "X *is isomorphic to* Y" if X and Y are groups. "$X \approx_{PL} Y$" means "X *is* PL *homeomorphic to* Y." If X is a subset of a manifold, $N_\varepsilon(X)$ denotes the open ε-neighborhood of X.

2. A CRITERION FOR WEAK FLATNESS

THEOREM 2.1. *Suppose* $\Sigma^k \subset S^n$ *is a topologically embedded* k-*sphere* ($n \geq 5$, $2 \leq k \leq n - 3$). *Then* Σ^k *is weakly flat if and only if* $S^n - \Sigma^k$ *is* 1 - ℓc *at* Σ^k.

Received February 2, 1968.
Most of the results in this paper are contained in the author's doctoral dissertation, which was written under the direction of Professor C. H. Edwards, Jr. at the University of Georgia. The research in this paper was supported by a National Science Foundation Graduate Fellowship.

In the proof of Theorem 2.1, we shall use the following special case of L. C. Siebenmann's open-collar theorem [10].

Suppose W^n $(n \geq 5)$ *is a connected* PL *manifold such that* $\partial W \subset W$ *is a homotopy equivalence,* W *is* 1-*connected at infinity, and* $\pi_1(\partial W) = 0$. *Then* $W \approx_{PL} \partial W \times [0, 1)$.

Proof of Theorem 2.1. The necessity follows from the easily verified fact that with our dimensional restrictions, $S^{n-k-1} \times R^{k+1}$ is 1-connected at infinity. To prove the converse, suppose $S^n - \Sigma^k$ is $1 - \ell c$ at Σ^k, $2 \leq k \leq n - 3$, and $n \geq 5$.

1) There is a family $\mathcal{M}$ of n-dimensional PL submanifolds of S^n such that each member of $\mathcal{M}$ contains Σ^k in its interior, each member of $\mathcal{M}$ has 1-connected boundary, and if U is an open set containing Σ^k, then there exists an $M \in \mathcal{M}$ such that $M \subset U$.

To prove 1), let U be any neighborhood of Σ^k, and let $V \subset U$ be a neighborhood of Σ^k such that each loop in $V - \Sigma^k$ is null homotopic in $U - \Sigma^k$. Using the regular-neighborhood theorem, one can find a PL submanifold M_1^n such that

$$\Sigma^k \subset \operatorname{int} M_1 \subset M_1 \subset V.$$

Since Σ^k does not separate any open set, we may assume that ∂M_1 is connected. (This is a standard hole-boring argument.) Now ∂M_1 may not be simply connected, but each loop in ∂M_1 is null homotopic in $U - \Sigma^k$; therefore we can do surgery on ∂M_1 to get a manifold M^n with 1-connected boundary such that $\Sigma^k \subset \operatorname{int} M \subset M \subset U$. (For details of such an argument, see [2].) M is the member of $\mathcal{M}$ corresponding to U.

Now choose any $M \in \mathcal{M}$. By the theorem of Van Kampen, M is 1-connected. Because $M - \Sigma^k$ is $1 - \ell c$ at Σ^k, we can choose a connected open neighborhood A of Σ^k in int M such that the inclusion $(A - \Sigma^k) \subset (M - \Sigma^k)$ induces the trivial map on fundamental groups. Since $M = (M - \Sigma^k) \cup A$ and the set $(M - \Sigma^k) \cap A = A - \Sigma^k$ is connected, the Van Kampen theorem implies that there is a commutative diagram

$$\begin{array}{ccccc} & & \pi_1(A - \Sigma^k) & & \\ & \swarrow & & \searrow & \\ \pi_1(A) & \longrightarrow & 0 & \longleftarrow & \pi_1(M - \Sigma^k) \\ & {\scriptstyle 0}\searrow & \downarrow & \swarrow{\scriptstyle \text{identity}} & \\ & & \pi_1(M - \Sigma^k). & & \end{array}$$

It follows that 2) $\pi_1(M - \Sigma^k) = 0$ for each $M \in \mathcal{M}$.

Using 2) and the Van Kampen theorem, we get 3) $\pi_1(S^n - \Sigma^k) = 0$.

Now we proceed as in [10]. Alexander Duality implies that $S^n - \Sigma^k$ has the homology groups of S^{n-k-1}, and by the Hurewicz theorem together with 3),

$$\pi_j(S^n - \Sigma^k) \approx \begin{cases} 0 & (j < n - k - 1), \\ Z & (j = n - k - 1). \end{cases}$$

By Irwin's theorem [13, Theorem 23, Chapter 8], there exists a PL sphere $\Sigma^{n-k-1} \subset S^n - \Sigma^k$ that represents a generator of $\pi_{n-k-1}(S^n - \Sigma^k)$, and by a theorem of Whitehead [12], the inclusion $\Sigma^{n-k-1} \subset S^n - \Sigma^k$ is a homotopy equivalence. Let

N be a regular neighborhood of Σ^{n-k-1} in $S^n - \Sigma^k$. Since $n - k - 1 \leq n - 3$, it follows from [13] that $(S^n, \Sigma^{n-k-1}) \approx_{PL} (S^n, S^{n-k-1})$. Thus Σ^{n-k-1} has a PL product neighborhood and $N \approx_{PL} S^{n-k-1} \times D^{k+1}$, by the uniqueness of regular neighborhoods.

Let $W = (S^n - \Sigma^k) - \text{int } N$. Then W is a connected PL n-manifold, $\partial W = \partial N$ is 1-connected, and by excision and the Whitehead Theorem, $\partial W \subset W$ is a homotopy equivalence. 2) implies that W is 1-connected at infinity; therefore $W \approx_{PL} \partial N \times [0, 1)$, by the open-collar theorem. Thus

$$S^n - \Sigma^k \approx_{PL} \text{int } N \approx_{PL} S^{n-k-1} \times R^{k+1};$$

consequently, Σ^k is weakly flat, and the proof is complete.

Remark. As the referee has pointed out, our argument actually proves the following stronger theorem.

If $X \subset S^n$ *is a compact* ANR *that is a homology* k-*sphere* $(n \geq 5,\ 2 \leq k \leq n - 3)$, *then* $S^n - X \approx S^{n-k-1} \times R^{k+1}$ *if and only if* $S^n - X$ *is* $1 - \ell c$ *at* X.

Beginning with $S^0 = \{-1, +1\} \subset R^1$, we may think of $S^n \subset R^{n+1}$ as the join $S^{n-1} * \{(0, 0, \cdots, 0, -1), (0, 0, \cdots, 0, 1)\}$. Given a sphere-pair (S^{n-1}, Σ^{k-1}), define the *suspension* $\text{Susp}(S^{n-1}, \Sigma^{k-1})$ to be the pair

$$(S^n, \Sigma^{k-1} * \{(0, 0, \cdots, 0, -1), (0, 0, \cdots, 0, 1)\}).$$

Let us say that (S^n, Σ^k) is a *suspension pair* if there exists a sphere-pair (S^{n-1}, Σ^{k-1}) such that $(S^n, \Sigma^k) \approx \text{Susp}(S^{n-1}, \Sigma^{k-1})$.

THEOREM 2.2. *Let* (S^n, Σ^k) *be a suspension pair* $(n \geq 5,\ 1 \leq k \leq n - 3)$. *Then* Σ^k *is weakly flat if and only if* $S^n - \Sigma^k$ *is simply connected.*

Proof. The necessity is obvious. For the sufficiency, let $S^n - \Sigma^k$ be simply connected, and let (S^{n-1}, Σ^{k-1}) be a pair such that $(S^n, \Sigma^k) \approx \text{Susp}(S^{n-1}, \Sigma^{k-1})$. Clearly, $S^n - \Sigma^k \approx (S^{n-1} - \Sigma^{k-1}) \times R^1$ and $S^{n-1} - \Sigma^{k-1}$ is simply connected.

If $k = 1$, then $S^{n-1} - \Sigma^0 \approx S^{n-2} \times R^1$; therefore $(S^{n-1} - \Sigma^0) \times R^1 \approx S^{n-2} \times R^2$ and Σ^1 is weakly flat.

If $k > 1$, then $S^{n-1} - \Sigma^{k-1}$ has one end, and by a theorem of J. Stallings [11, Proposition 2.2], $(S^{n-1} - \Sigma^{k-1}) \times R^1$ is 1-connected at infinity. This implies that $S^n - \Sigma^k$ is $1 - \ell c$ at Σ^k, so that Σ^k is weakly flat, by Theorem 2.1.

3. A REDUCTION THEOREM

McMillan [8] has shown that if X is a polyhedral AR (absolute retract) and $h: X \to M^n$ is an embedding of X in a PL manifold such that $M^n - h(X)$ is $1 - \ell c$ at h(X), then $M^n - h(Y)$ is $1 - \ell c$ at h(Y), for each subpolyhedron $Y \subset X$ such that X collapses to Y. Since we are interested in embeddings of closed manifolds, we need a different reduction.

If A is a space, we define the *cone* CA to be the quotient space $A \times [0, 1]/(A \times \{0\})$, and we denote by $p: A \times [0, 1] \to CA$ the quotient map. For $0 < s < r \leq 1$, define

$$CA_r = p(A \times [0, r]), \qquad A_r = p(A \times \{r\}), \qquad [A_s, A_r] = p(A \times [s, r]).$$

We make the natural identification $A_1 = A$.

THEOREM 3.1. *Let* X^k *be a* 1-*connected* ANR, *and suppose that* $X^k = Y \cup Z$, *where* Y *and the set* $A = Y \cap Z$ *are compact*, 1-*connected* ANR'*s and* $(Z, A) \approx (CA, A)$. *Suppose* $h: X \to S^n$ *is an embedding* $(n \geq 5,\ k \leq n-2)$ *and* $S^n - h(X)$ *is* $1 - \ell c$ *at* $h(X)$. *Then* $S^n - h(Y)$ *is* $1 - \ell c$ *at* $h(Y)$.

Proof. We shall identify (Z, A) with (CA, A), so that we can use the above notation. Let U be any neighborhood of h(Y). Choose an r $(0 < r < 1)$ such that

$$(h(X) - h(CA_r)) \cup h(A_r) \subset U.$$

Choose η $(0 < \eta < r)$ so that $h([A_{r-\eta}, A_{r+\eta}]) \subset U - h(Y)$. By a standard argument, the fact that $h([A_{r-\eta}, A_{r+\eta}])$ is a 1-connected ANR implies the existence of an open set $Q \subset U - h(Y)$ such that $h([A_{r-\eta}, A_{r+\eta}]) \subset Q$ and each loop in Q is null homotopic in U - h(Y). Let M^n be a PL submanifold of S^n such that

$$h([A_{r-\eta}, A_{r+\eta}]) \subset \text{int } M^n \subset M^n \subset Q.$$

Let $\varepsilon > 0$ be so small that

1) $N_\varepsilon(h(Y \cup [A_{r+\eta}, A])) \subset U$,

2) $N_\varepsilon(h([A_{r-\eta}, A_{r+\eta}])) \subset \text{int } M^n$, and

3) $N_\varepsilon(h(Y \cup [A_{r+\eta}, A])) \cap N_\varepsilon(h(CA_{r-\eta})) = \emptyset$.

Notice that by 1), 2), and 3), every connected subset of $N_\varepsilon(h(X))$ that intersects $N_\varepsilon(h(Y \cup [A_{r+\eta}, A]))$ and misses int M^n is contained in U.

Since $S^n - h(X)$ is $1 - \ell c$ at h(X), there exists a $\delta > 0$ $(0 < \delta < \varepsilon)$ such that each loop in $N_\delta(h(X)) - h(X)$ is null homotopic in $N_\varepsilon(h(X)) - h(X)$. Let V be the component of $N_\delta(h(Y \cup [A_{r+\eta}, A])) - M$ containing h(Y). Then $V \subset U$, and we claim that each loop in V - h(Y) is null homotopic in U - h(Y).

To see this, let $f: \partial D^2 \to V - h(Y)$ be any map. Since h(X) has codimension at least two, we may assume that $f(\partial D^2) \cap h(X) = \emptyset$. By simplicial approximation and general position, we may assume that f is a PL embedding. Since f is a loop in $N_\delta(h(X)) - h(X)$, f extends to a map $F: D^2 \to N_\varepsilon(h(X)) - h(X)$. By general position, we may assume that F is a PL embedding and that $F(D^2) \cap \partial M$ is either empty or consists of a finite number of disjoint simple closed curves. (This is a standard general-position argument; see for example Edwards [5, Lemma 2].) If $F(D^2) \cap \partial M = \emptyset$, our remarks above imply that $F(D^2) \subset U$, so that f is null homotopic in U - h(Y). Otherwise, let $C_1, \cdots, C_q$ be the components of $F^{-1}(\partial M)$ that are not in the interior of any other component of $F^{-1}(\partial M)$. Then, as above,

$$F\left(D^2 - \bigcup_{i=1}^{q} \text{int } C_i\right) \subset U - h(Y),$$

and by our choice of W, we can redefine F on $\bigcup_{i=1}^{q} \text{int } C_i$ to get a map $G: D^2 \to U - h(Y)$ that extends f. Therefore f is null homotopic in U - h(Y), and the proof is complete.

Suppose X is an ANR that can be written as a union $Y \cup B^k$, where B^k is a k-cell and $Y \cap B^k = \partial B^k$. Then we say that Y can be obtained from X by a *perforation of order* k.

COROLLARY 3.2. *Suppose* X^k *is a* 1*-connected* ANR, *and* Y *can be obtained from* X *by a finite sequence of perforations of order at least* 3. *If* $h: X^k \to S^n$ $(n \geq 5,\ k \leq n - 2)$ *is an embedding such that* $S^n - h(X)$ *is* $1 - \ell c$ *at* $h(X)$, *then* $S^n - h(Y)$ *is* $1 - \ell c$ *at* $h(Y)$.

4. APPLICATIONS

THEOREM 4.1. *If* $\Sigma^k \subset S^n$ *bounds a cellular* $(k+1)$*-cell in* S^n $(n \geq 5,\ 2 \leq k \leq n - 3)$, *then* Σ^k *is weakly flat.*

Proof. Suppose Σ^k bounds the cellular $(k+1)$-cell B^{k+1} in S^n. Then $S^n - B^{k+1}$ is $1 - \ell c$ at B^k, by [7]. Applying Theorem 3.1 with $X = D^{k+1}$ and $Y = A = S^k$, we find that $S^n - \Sigma^k$ is $1 - \ell c$ at Σ^k, so that Σ^k is weakly flat, by Theorem 2.1.

In the proof of Theorem 4.3, we shall need the following lemma, which is an immediate consequence of Brown's monotone-union theorem [3].

LEMMA 4.2. *If* $X \subset S^n$ *is an intersection* $\bigcap_{i=1}^{\infty} X_i$, *where each* X_i *is cellular and* X_{i+1} *is properly contained in* X_i *for each* i, *then* X *is cellular.*

THEOREM 4.3. *If* $B^k \subset S^n$ *is a* k*-cell and* B^k *is cellular in* S^n $(n \geq 5,\ 1 \leq k \leq n - 2)$, *then every set that is cellular in* int B^k *is cellular in* S^n.

Proof. Let $h: D^k \to B^k$ be a homeomorphism onto. Let $X \subset \text{int } D^k$ be a cellular subset. We want to show that $h(X)$ is cellular. Since X is cellular, $D^k - X \approx S^{k-1} \times [0, 1)$. Therefore we can write X as an intersection $\bigcap_{i=1}^{\infty} D_i$, where for each i,

$$D_i \text{ is a } k\text{-cell}, \quad D_{i+1} \subset \text{int } D_i, \quad D^k - \text{int } D_i \approx \partial D_i \times [0, 1].$$

For a fixed i, let C_1 and C_2 be $(k-1)$-cells such that $\partial D_i = C_1 \cup C_2$ and $C_1 \cap C_2 = \partial C_1 = \partial C_2$. Then

$$D^k \approx D_i \cup (C_1 \times [0, 1]) \cup (C_2 \times [0, 1]),$$

where we identify C_j with $C_j \times \{0\}$ (j = 1, 2), and where

$$(C_1 \times [0, 1]) \cap (C_2 \times [0, 1]) = \partial C_2 \times [0, 1].$$

Since $C_2 \times [0, 1]$ and $D_i \cup (C_1 \times [0, 1])$ intersect in the common $(k-1)$-cell $C_2 \cup (\partial C_2 \times [0, 1])$, Theorem 3.1 shows that $S^n - h(D_i \cup (C_1 \times [0, 1]))$ is $1 - \ell c$ at $h(D_i \cup (C_1 \times [0, 1]))$. A second application of Theorem 3.1 shows that $S^n - h(D_i)$ is $1 - \ell c$ at $h(D_i)$; therefore, by the cellularity criterion [7], $h(D_i)$ is cellular in S^n. Since $h(X) = \bigcap_{i=1}^{\infty} h(D_i)$, Lemma 4.2 implies that $h(X)$ is cellular in S^n.

Remark. If $k \neq 4$, then each cellular subset of D^k is cellular with respect to PL cells [7], [11]; therefore Theorem 4.3 also follows from McMillan's collapsing theorem [8] and the PL-annulus theorem.

THEOREM 4.4. *If* $\Sigma^k \subset S^n$ *is weakly flat* $(3 \leq k \leq n - 3)$, *then each set that is cellular in* Σ^k *is cellular in* S^n.

Proof. Since Σ^k is weakly flat, $S^n - \Sigma^k$ is $1 - \ell c$ at Σ^k. Let $X \subset \Sigma^k$ be cellular. Then there exists a flat (relative to Σ^k) k-cell $B^k \subset \Sigma^k$ such that $X \subset \text{int } B^k$

and X is cellular in int B^k. Because B^k is cellular in S^n, by Corollary 3.2 and McMillan's cellularity criterion [7], it follows from Theorem 4.3 that X is cellular in S^n

In the light of Theorem 4.3, one might ask about the transitivity of weak flatness. That is, if $\Sigma^k \subset \Sigma^m \subset S^n$, with Σ^k weakly flat in Σ^m and Σ^m weakly flat in S^n, under what additional conditions is Σ^k weakly flat in S^n? We shall need the following lemma.

LEMMA 4.5. *Suppose that* $2 \leq k < m$ *and* T *is homeomorphic to* $S^{m-k-1} \times D^{k+1}$. *Then* ∂T *can be obtained from* T *by a finite sequence of perforations of order at least three.*

Proof. Let $q = m - k$. The proof is by induction on q. If $q = 1$, then $T \approx A_1 \cup A_2$ where A_1 and A_2 are disjoint $(k+1)$-cells. Since $k + 1 \geq 3$, the lemma holds in this case.

Suppose then that $q > 1$ and that the lemma holds for each positive integer less than q. Write S^{q-1} as $B_1 \cup B_2$, where each B_i is a $(q - 1)$-cell and $B_1 \cap B_2 = \partial B_1 = \partial B_2$. Then

$$T \approx (B_1 \times D^{k+1}) \cup (B_2 \times D^{k+1}) = A_1 \cup A_2,$$

where $A_i = B_i \times D^{k+1}$. Let $T^* = A_1 \cap A_2$. Clearly, $T^* \approx S^{q-2} \times D^{k+1}$. Also, $T^* \cap \partial T = \partial T^*$ and $T - (\text{int } A_1 \cup \text{int } A_2) = \partial T \cup T^*$. Since we can obtain $T - (\text{int } A_1 \cup \text{int } A_2)$ from T by two perforations of order at least three, the lemma follows by induction.

Notice that if $T \subset S^n$ is homeomorphic to $S^{m-k-1} \times D^{k+1}$ and $W = S^m - \text{int } T$ (with m and k as above), then W can be obtained from S^m by a finite sequence of perforations of order at least three, by Lemma 4.5.

THEOREM 4.6. *Suppose that* $\Sigma^k \subset \Sigma^m \subset S^n$ $(2 \leq k < m \leq n - 3)$, *that* Σ^k *is weakly flat in* Σ^m, *and that* Σ^m *is weakly flat in* S^n. *Then* Σ^k *is weakly flat in* S^n.

Proof. Since Σ^k is weakly flat in Σ^m, we can represent Σ^k as an intersection $\bigcap_{i=1}^{\infty} W_i$, where each W_i is the closure of the complement of a locally flat copy of $S^{m-k-1} \times D^{k+1}$ in Σ^m, and where $\Sigma^k \subset \text{int } W_{i+1} \subset \text{int } W_i$ for each i. By the remark after Theorem 4.2 and by Corollary 3.2, $S^n - W_i$ is $1 - \ell c$ at W_i for each i. Let U be any open neighborhood of Σ^k in S^n. For a sufficiently large i, W_i is contained in U. Let $V \subset U$ be a neighborhood of W_i such that each loop in $V - W_i$ is null homotopic in $U - W_i$. Then each loop in $V - \Sigma^k$ is homotopic in $V - \Sigma^k$ to a loop in $V - W_i$, because W_i has codimension at least three in S^n. Therefore each loop in $V - \Sigma^k$ is null homotopic in $U - \Sigma^k$. Thus $S^n - \Sigma^k$ is $1 - \ell c$ at Σ^k, and Σ^k is weakly flat in S^n.

5. AN EXAMPLE

In this section, we give an example to show that a sphere can be embedded very badly and still be weakly flat. Given $\Sigma^k \subset S^n$ and $x \in \Sigma^k$, we say that Σ^k is *locally flat* at x if x has a neighborhood U in S^n such that $(U, U \cap \Sigma^k) \approx (R^n, R^k)$. We say that Σ^k is *locally nice* at x if for each neighborhood U of x in S^n there exists a smaller neighborhood V of x such that each loop in $V - \Sigma^k$ is null homotopic in $U - \Sigma^k$.

THEOREM 5.1. *For* $n \geq 6$, *there exists a weakly flat* $(n - 3)$*-sphere in* S^n *that is not locally flat at any point and is locally nice at exactly one point.*

Proof. Let A be a Fox-Artin arc in R^3 that is not cellular but whose complement in R^3 is simply connected (see Example 1.3 of [6]). By a theorem of Andrews and Curtis [1], $R^3/A \times R^{n-3} \approx R^n$. Identify S^n with the one-point compactification $(R^3/A \times R^{n-3}) \cup \{q\}$, and set $\Sigma^{n-3} = (\{a\} \times R^{n-3}) \cup \{q\}$, where $a \in R^3/A$ is the image of A under the quotient map.

Since $S^n - \Sigma^{n-3} \approx (R^3 - A \times R^{n-3})$, and since the right-hand member is 1-connected at infinity (see Stallings [11]), Theorem 2.1 implies that Σ^{n-3} is weakly flat.

Since A is not cellular in R^3, the cellularity criterion implies that there exists a neighborhood U of A such that for every smaller neighborhood V of A there is a loop in V - A that is essential in U - A. Using this and the product structure in $R^3/A \times R^{n-3}$, we can easily show that Σ^{n-3} is not locally nice at any point of $\{a\} \times R^{n-3}$. Since local flatness in codimension three implies local niceness, Σ^{n-3} is not locally flat at any point of $\{a\} \times R^{n-3}$. Because the set of points at which an embedded sphere fails to be locally flat is closed, Σ^{n-3} is not locally flat at any point.

To show that Σ^{n-3} is locally nice at q, it suffices to show that there exist arbitrarily large compact sets in $R^3/A \times R^{n-3}$ of the form $Y \times B$ such that $S^n - ((Y \times B) \cup \Sigma^{n-3})$ is 1-connected. To construct such a set, let Y be the image in R^3/A of a large tame 3-ball containing A, and let B be a large tame $(n - 3)$-ball about the origin in R^{n-3}. Then $R^3/A - Y$ and $R^{n-3} - B$ are 1-connected and

$$S^n - ((Y \times B) \cup \Sigma^{n-3}) = [(R^3/A - Y) \times R^{n-3}] \cup [(R^3/A - \{a\}) \times (R^{n-3} - B)].$$

Therefore the theorem of Van Kampen implies that $S^n - ((Y \times B) \cup \Sigma^{n-3})$ is 1-connected, and Σ^{n-3} is locally nice at q.

For $r > 0$, let B_r denote the $(n - 3)$-ball in R^{n-3} with radius r and center at the origin. By means of the cellularity criterion it is easy to show that each ball in S^n of the form $\{a\} \times B_r$ is cellular in S^n, so that Σ^{n-3} may be written as the union of $\{q\}$ and a monotone union of cellular $(n - 3)$-cells. This illustrates the following theorem, which is proved in [4].

THEOREM 5.2. *Suppose* $\Sigma^k \subset S^n$ *is a* k*-sphere* $(n \geq 5,\ 2 \leq k \leq n - 3)$ *and* Σ^k *has the property that for some* $p \in \Sigma^k$, Σ^k *is locally nice at* p *and* $\Sigma^k - \{p\}$ *is the monotone union of cellular* k*-cells. Then* Σ^k *is weakly flat.*

If we alter the construction in Theorem 5.1 by choosing A to be an arc in R^3 whose complement is not simply connected, we get an $(n - 3)$-sphere Σ_1^{n-3} in S^n that is not weakly flat, because $S^n - \Sigma_1^{n-3}$ is not simply connected. However, one can show, as in Theorem 5.1, that $\Sigma_1^{n-3} - \{q\}$ is the monotone union of cellular $(n - 3)$-cells.

REFERENCES

1. J. J. Andrews and M. L. Curtis, n-*space modulo an arc.* Ann. of Math. (2) 75 (1962), 1-7.

2. W. Browder, *Structures on* $M \times R$. Proc. Cambridge Philos. Soc. 61 (1965), 337-345.

3. M. Brown, *The monotone union of open* n*-cells is an open* n*-cell.* Proc. Amer. Math. Soc. 12 (1961), 812-814.

4. P. F. Duvall, Jr., *Neighborhoods of* 1*-connected* ANR's *in high dimensional piecewise linear manifolds.* Doctoral Thesis, University of Georgia, 1967.

5. C. H. Edwards, Jr., *Taming* 2*-complexes in high-dimensional manifolds.* Duke Math. J. 32 (1965), 479-494.

6. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space.* Annals of Math. (2) 49 (1948), 979-990.

7. D. R. McMillan, Jr., *A criterion for cellularity in a manifold.* Ann. of Math. (2) 79 (1964), 327-337.

8. ———, *Piercing a disk along a cellular arc* (to appear).

9. R. H. Rosen, *The five dimensional polyhedral Schoenflies theorem.* Bull. Amer. Math. Soc. 70 (1964), 511-516.

10. L. C. Siebenmann, *On detecting open collars* (to appear).

11. J. Stallings, *The piecewise-linear structure of Euclidean space.* Proc. Cambridge Philos. Soc. 58 (1962), 481-488.

12. J. H. C. Whitehead, *On the homotopy type of* ANR's*.* Bull. Amer. Math. Soc. 54 (1948), 1133-1145.

13. E. C. Zeeman, *Seminar on combinatorial topology.* Mimeographed notes, Inst. Hautes Études Sci., Paris, 1963.

The University of Georgia
Athens, Georgia 30601