ACYCLICITY OF COMPACT MEANS

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The problem of determining the spaces that admit an n-mean was considered first by G. Aumann [2] and later by B. Eckmann [6], the major tool of the latter being homotopy and singular homology. In this paper, we use Alexander cohomology to establish a result analogous to that of Eckmann [6]; it enables us to expand the class of spaces known not to admit an n-mean. We are also able to extend to Euclidean n-space some results of P. Bacon [3], [4] concerning means in the plane.

1. PRELIMINARIES

An *n-mean* ($n \geq 2$) on a Hausdorff space X is a continuous function $\mu: X^n \to X$ satisfying the conditions

(i) $\mu(x, \cdots, x) = x$ for each x in X (idempotence);

(ii) $\mu(x_1, \cdots, x_n)$ is symmetric in its argument (symmetry).

A space equipped with a 2-mean is then simply an idempotent, commutative, topological groupoid. If, for a permutation σ of $\{1, 2, \cdots, n\}$, $T_\sigma: X^n \to X^n$ denotes the map

$$T_\sigma(x_1, \cdots, x_n) = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$$

and $\Delta: X \to X^n$ denotes the diagonal map $\Delta(x) = (x, \cdots, x)$, then conditions (i) and (ii) may be expressed as $\mu\Delta = 1_X$ and $\mu T_\sigma = \mu$, respectively.

We shall use Alexander cohomology, and for simplicity we shall assume that the coefficients are taken in a fixed principal ideal domain R, unless it is otherwise stated. An R-module A is *uniquely n-divisible* if for each x in A, there exists a unique x' in A such that nx' = x, in other words, if the function $x \mapsto nx$ is an automorphism of A. A *continuum* is a compact, connected Hausdorff space.

The proof of the main theorem of the paper is based on the following n-fold version of the Künneth formula for Alexander cohomology (see [8], especially Theorem 11 on p. 247, Corollary 2 on p. 312, and E2 and E6 on pp. 359, 360).

KÜNNETH FORMULA. *If X is a compact Hausdorff space, then there is a short exact sequence forming the rows of the diagram*

$$\begin{array}{ccccccc} 0 \to & \sum_{i_1+\cdots+i_n=p} H^{i_1}(X)\otimes\cdots\otimes H^{i_n}(X) & \xrightarrow{\lambda} & H^p(X^n) & \to & T^{p+1} & \to 0 \\ & \downarrow w_\sigma & & \downarrow T_\sigma^* & & & \\ 0 \to & \sum_{i_1+\cdots+i_n=p} H^{i_1}(X)\otimes\cdots\otimes H^{i_n}(X) & \xrightarrow{\lambda} & H^p(X^n) & \to & T^{p+1} & \to 0 \end{array}$$

and making it analytic; here σ denotes any permutation of $\{1, 2, \cdots, n\}$,

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$$w_\sigma(h^{i_1} \otimes \cdots \otimes h^{i_n}) = (-1)^{i_1 \cdots i_n}(h^{i_{\sigma(1)}} \otimes \cdots \otimes h^{i_{\sigma(n)}}),$$

and $T^{p+1} = \sum_{i_1+\cdots+i_n=p+1} H^{i_1}(X) * \cdots * H^{i_n}(X).$

The following is an immediate consequence of the identities

$$n(a \otimes b) = (na) \otimes b = a \otimes (nb) \quad \text{and} \quad n(a + b) = na + nb.$$

LEMMA. *Suppose that* A *and* B *are* R-*modules. If* A *or* B *is uniquely* n-*divisible, then so is* $A \otimes B$, *and if both* A *and* B *are uniquely* n-*divisible, then so is* $A \oplus B$.

2. THE RESULTS

THEOREM. *Suppose that* X *is a continuum such that* $H^p(X)$ *is torsion-free (as an* R-*module) for each* $p \geq 0$. *If* X *admits an* n-*mean, then* $H^p(X)$ *is uniquely* n-*divisible for each* $p \geq 1$.

Proof. The hypothesis that $H^p(X)$ is torsion-free ensures that the λ of the Künneth formula is an isomorphism. Since X is connected, $H^0(X) \cong R$, so that

$$H^{i_1}(X) \otimes \cdots \otimes H^{i_n}(X) \cong H^p(X)$$

whenever $i_k = 0$ for all except one index k. Taking note of this, and renaming w_σ and λ accordingly, we obtain the analytic diagram

$$\begin{array}{ccc} H^p(X) \times \cdots \times H^p(X) \times \Sigma^p & \xrightarrow{\lambda} & H^p(X^n) \\ \Big\downarrow w_\sigma & & \Big\downarrow T_\sigma^* \\ H^p(X) \times \cdots \times H^p(X) \times \Sigma^p & \xrightarrow{\lambda} & H^p(X^n) \end{array},$$

where

$$\Sigma^p = \sum_{\substack{i_1+\cdots+i_n=p \\ 0 \leq i_1, \cdots, i_n < p}} H^{i_1}(X) \otimes \cdots \otimes H^{i_n}(X)$$

and $w_\sigma(h_1, \cdots, h_n, h_*) = (h_{\sigma(1)}, \cdots, h_{\sigma(n)}, h_*')$ for some h_*' in Σ^p.

We show first that if Σ^p is uniquely n-divisible, then so is $H^p(X)$. Let h be an element of $H^p(X)$, and let $\mu^*(h) = \lambda(h_1, \cdots, h_n, h_*)$, where μ is an n-mean on X. Since $\mu^* = T_\sigma^* \mu^*$ for each permutation σ, we obtain the equation

$$\lambda(h_1, \cdots, h_n, h_*) = \lambda(h_{\sigma(1)}, \cdots, h_{\sigma(n)}, h_*')$$

for each permutation σ, so that $h_1 = h_2 = \cdots = h_n$.

The idempotency of the mean implies that $\Delta^* \mu^*$ is the identity on $H^p(X)$, and therefore

$$h = \Delta^* \mu^*(h) = \Delta^* \lambda(h_1, \cdots, h_1, h_*)$$
$$= n\Delta^* \lambda(h_1, 0, \cdots, 0) + n\Delta^* \lambda(0, \cdots, 0, (1/n)h_*)$$
$$= n\Delta^* \lambda(h_1, 0, \cdots, 0, (1/n)h_*).$$

It follows that if we define $\rho: H^p(X) \to H^p(X)$ by $\rho = \Delta^* \lambda s \lambda^{-1} \mu^*$, where $s(h_1, \cdots, h_n, h_*) = (h_1, 0, \cdots, 0, (1/n)h_*)$, then $h = n\rho(h) = \rho(nh)$. Hence $H^p(X)$ is uniquely n-divisible.

By observing that Σ^1 is trivial and appealing to the lemma, one easily deduces the theorem by an inductive argument.

By taking integral coefficients, one sees that no cohomological p-sphere $(p \geq 1)$ can admit an n-mean, since $H^p(S^p) \cong Z$ is not n-divisible (see Section 3).

The theorem enables us to extend the results of Bacon [3] on n-means in the plane to R^p (Corollaries 1 and 2) and to obtain his unicoherence result in [4] for n-means (Corollary 3). We note, however, that in some of Bacon's results a weaker form of symmetry than ours is assumed.

COROLLARY 1. *If* X *is a compact space admitting an* n-*mean, then* $H^p(X; Z_n) = 0$ *for all* $p \geq 1$.

Proof. Suppose first that X is connected and that q is a prime dividing n. Then Z_q is a field such that $H^p(X; Z_q)$ is a vector space and hence is torsion-free for all $p \geq 0$. It follows from the theorem that $H^p(X; Z_q)$ is n-divisible. But each element of $H^p(X; Z_q)$ is of additive order q; therefore $H^p(X; Z_q) = 0$. Now let $n = q_1 q_2 \cdots q_k$, where the q_i are primes. According to a result of Gordon [7], the short exact sequence

$$0 \to Z_{q_1 \cdots q_r} \to Z_{q_1 \cdots q_{r+1}} \to Z_{q_{r+1}} \to 0$$

induces an exact sequence

$$H^p(X; Z_{q_1 \cdots q_r}) \to H^p(X; Z_{q_1 \cdots q_{r+1}}) \to H^p(X; Z_{q_{r+1}}).$$

A simple inductive argument then gives the equation $H^p(X; Z_n) = 0$. (Indeed, $H^p(X; Z_m) = 0$, provided each prime divisor of m is a prime divisor of n.)

If X is not connected, then each component C of X admits an n-mean [2, Satz 4], so that $H^p(C; Z_n) = 0$ (by the first part of the proof). But if $H^p(C; Z_n) = 0$ for each component C of X, then $H^p(X; Z_n) = 0$ [9, p. 44].

COROLLARY 2. *If a compact subset of* R^p $(p \geq 2)$ *admits an* n-*mean, then it does not cut* R^p.

Proof. If X is such a subset, then $H^{p-1}(X; Z_n) = 0$, by Corollary 1. The cohomological version of Bacon's proof of (2.3) in [3] now implies that X does not cut R^p.

COROLLARY 3. *A continuum that admits an* n-*mean is unicoherent.*

Proof. For such a space X, we know from Corollary 1 that $H^1(X; Z_n) = 0$. Suppose that A and B are subcontinua of X such that $X = A \cup B$, and consider the absolute Mayer-Vietoris sequence, reduced in dimension zero:

$$\rightarrow \tilde{H}^0(A; Z_n) \times \tilde{H}^0(B; Z_n) \rightarrow \tilde{H}^0(A \cap B; Z_n) \rightarrow H^1(X; Z_n) \rightarrow .$$

Since connectivity of a space is equivalent to the vanishing of its reduced zero-dimensional cohomology group, we see that $\tilde{H}^0(A; Z_n) = 0 = \tilde{H}^0(B; Z_n)$ and hence that $\tilde{H}^0(A \cap B; Z_n) = 0$. Thus $A \cap B$ is connected and X is unicoherent.

The next corollary shows that, up to the unsolved problem whether codimension depends on the coefficients, the assumption of associativity in the paper of L. W. Anderson and L. E. Ward [1] is unnecessary.

COROLLARY 4. *If* X *is a locally connected continuum admitting an* n*-mean and having codimension* $(X; Z_n) = 1$, *then* X *is a tree.*

Proof. Suppose A is a subcontinuum of X. The assumption that codimension $(X; Z_n) = 1$ implies $H^2(X, A; Z_n) = 0$ [5, Theorem 3.3], and Corollary 1 implies that $H^1(X; Z_n) = 0$. Appealing to the exact sequence

$$\rightarrow H^1(X; Z_n) \rightarrow H^1(A; Z_n) \rightarrow H^2(X, A; Z_n) \rightarrow ,$$

we conclude that $H^1(A; Z_n) = 0$. As in the proof of Corollary 3, we can show that A is unicoherent. But Ward [10, Theorem 9] has shown that a locally connected continuum all of whose subcontinua are unicoherent is a tree.

3. EXAMPLES

Set $r = 1/\pi$, and consider the following subsets of R^3:

$$A = \{(x, y, z) \mid z = \sin(x^2+y^2)^{-1/2},\ 0 < x^2+y^2 \leq r^2\},$$

$$B = \{(0, 0, z) \mid -1 \leq z \leq 1\},$$

$$C = \{(x, y, z) \mid x^2+y^2 = r^2,\ 0 \leq z \leq 2\},$$

$$D = \{(x, y, z) \mid x^2+y^2 \leq r^2,\ z = 2\}.$$

The space $X = A \cup B \cup C \cup D$ is a cohomological 2-sphere and hence admits no n-mean. However, its positive-dimensional homotopy and singular homology groups vanish, so that the results of Eckmann [6] do not apply. More generally, for each $p \geq 1$, one can compactify R^p by adding a real arc in such a way that the compactification is a cohomological p-sphere with vanishing positive-dimensional homotopy and singular homology groups.

The role of the coefficients in the foregoing results is illustrated by the n-adic solenoids Σ_n [8, p. 358]. The natural topological group structure on Σ_n has the property that $s(x) = x^n$ is an automorphism, so that $\mu(x_1, \cdots, x_n) = s^{-1}(x_1 \cdots x_n)$ defines an n-mean on Σ_n. In agreement with Corollary 1 we find that $H^1(\Sigma_2; Z_2) = 0$, while $H^1(\Sigma_2; Z)$, being isomorphic to the dyadic rationals, is nonzero but uniquely 2-divisible, as assured by the theorem. However, $H^1(\Sigma_2; Z)$ is not 3-divisible, and therefore Σ_2 admits no 3-mean.

We make a final remark regarding when the existence of an n-mean on a space implies the existence of an m-mean on the space. A moment's thought shows that a space admits an mn-mean if and only if it admits both an m-mean and an n-mean. Letting P(n) denote the set of prime divisors of n, we deduce that if a space admits an n-mean, then it admits an m-mean whenever $P(m) \subset P(n)$. The n-adic solenoids furnish examples to show that in general this is the most that can be said. For suppose that q is a prime divisor of m but not of n. Then $H^1(\Sigma_n; Z_q)$ is isomorphic to Z_q, so that Σ_n admits no m-mean while it does admit an n-mean.

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