UNIVALENT FUNCTIONS f(z) FOR WHICH zf'(z) IS SPIRALLIKE

M. S. Robertson

Let $\mathscr{F}$ denote the class of functions F(z) that are regular, univalent, and spirallike in the unit disk $E = \{z: |z| < 1\}$ and that are normalized so that $F(0) = 0$ and $F'(0) = 1$. L. Špaček [5] showed that these functions are characterized by the condition that for some real constant α ($|\alpha| < \pi/2$),

$$\Re\left\{e^{i\alpha}\frac{zF'(z)}{F(z)}\right\} > 0 \qquad (z \in E).$$

We denote the corresponding subclasses of $\mathscr{F}$ by $\mathscr{F}_\alpha$; in particular, $\mathscr{F}_0$ is the class of starlike functions. If $F(z) \in \mathscr{F}_0$, then the function $f(z) = \int_0^z \frac{F(t)}{t}dt$ maps E onto a convex domain, and

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) = \Re\frac{z[zf'(z)]'}{zf'(z)} = \Re\frac{zF'(z)}{F(z)} > 0 \qquad (z \in E).$$

In this note, we consider another family of functions that includes the class of convex functions as a proper subfamily. For $-\pi/2 < \alpha < \pi/2$, we say that $f(z) \in S_\alpha$ provided

(i) f(z) is regular in E, $f(0) = 0$, and $f'(0) = 1$,

(ii) $f'(z) \neq 0$ in E,

(iii) $\Re\left(e^{i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right) > 0 \qquad (z \in E).$

We note that the three conditions are precisely the conditions for the function $zf'(z)$ to belong to the class $\mathscr{F}_\alpha$. The class S_0 consists of the normalized convex functions.

For general values α ($-\pi/2 < \alpha < \pi/2$), a function in S_α need not be univalent in E. For example, the function

$$f(z) = i(1-z)^i - i = z + \cdots$$

belongs to the class $S_{\pi/4}$, but it has a zero at each of the points $1 - e^{-2n\pi}$ ($n = 0, 1, \cdots$), and in fact it assumes every value lying on the circle $|w+i| = 1$ infinitely often on the open segment (0, 1) of the real axis. (J. Krzyż and Z. Lewandowski were the first to point out that if $zf'(z)$ is spirallike, the function f(z) is not necessarily univalent; see [2].) However, we shall show that for a certain set of values of α, all functions in S_α are univalent in E.

The situation is analogous to a problem recently considered by P. L. Duren, H. S. Shapiro, and A. L. Shields [1] and by W. C. Royster [4]: for what values of a complex constant α is the function

Received August 2, 1968.
The author acknowledges support from the National Science Foundation (NSF-GP-7439).

$$g_\alpha(z) = \int_0^z [g'(t)]^\alpha \, dt$$

univalent in E whenever $g(z)$ is regular and univalent in E and normalized so that $g(0) = 0$, $g'(0) = 1$?

THEOREM. *Let* $f(z) \in S_\alpha$, *where* $0 < \cos\alpha \le x_0$ *and where* x_0 *denotes the positive root* $0.2315 \cdots$ *of the equation*

$$16x^3 + 16x^2 + x - 1 = 0.$$

Then $f(z)$ *is univalent in* E.

If $\mu + 1 = |\mu + 1| e^{-i\alpha}$ $(1/2 < \cos\alpha < 1)$, *and if moreover*

$$|\mu| \le 1, \qquad |\mu + 1| > 1, \qquad |\mu - 1| > 1,$$

then the function $f_\alpha^*(z) = \frac{1}{\mu}[(1-z)^{-\mu} - 1] = z + \cdots$ *belongs to* S_α *but is not univalent in* E.

LEMMA 1. *If* $f_0(z) = z + c_2 z^2 + \cdots$ *is regular and univalent in* E *and maps* E *onto a convex domain, then*

$$|w(f_0, z)| = \left| \left(\frac{f_0''(z)}{f_0'(z)} \right)' - \frac{1}{2}\left(\frac{f_0''(z)}{f_0'(z)} \right)^2 \right| \le \frac{2}{(1 - |z|^2)^2} \qquad (z \in E),$$

with equality for

$$f_0(z) = \frac{1}{2} \log \frac{1+z}{1-z}.$$

Proof. Let

$$1 + \frac{z f_0''(z)}{f_0'(z)} = P(z) = 1 + p_1 z + p_2 z^2 + \cdots.$$

Then $\Re P(z) > 0$ in E, and $|p_1| \le 2$, $|p_2| \le 2$. Since also $\Re [P(z)]^{-1} > 0$ in E and

$$[P(z)]^{-1} = 1 - p_1 z + (p_1^2 - p_2) z^2 + \cdots,$$

we have the inequality $|p_1^2 - p_2| \le 2$. But $c_2 = p_1/2$ and $c_3 = (p_2 + p_1^2)/6$. Therefore

$$|c_2^2 - c_3| = \frac{1}{12} |p_1^2 - 2p_2| \le \frac{1}{12} \{ |p_1^2 - p_2| + |p_2| \} \le 1/3.$$

Let $h(z)$ be the reciprocal of a convex function, and let it have the representation $h(z) = \frac{1}{z} + b_0 + b_1 z + \cdots$. If $h(z) = [f_0(z)]^{-1}$, then $|b_1| = |c_2^2 - c_3| \le 1/3$. Let x be a point of E, and define

$$g(z) = \frac{f_0'(x)(1 - |x|^2)}{f_0\left(\frac{x+z}{1+\bar{x}z}\right) - f_0(x)} = \frac{1}{z} + A_0(x) + A_1(x) z + \cdots.$$

Then, since $f_0\left(\frac{x+z}{1+\bar{x}z}\right)$ maps E onto a convex domain, g(z) is the reciprocal of a convex function, and we let h(z) = g(z). Now

$$\frac{1}{6}|w(f_0, x)|(1-|x|^2)^2 = |A_1(x)| \le 1/3,$$

that is,

$$|w(f_0, z)| \le 2(1-|z|^2)^{-2} \qquad (z \in E).$$

LEMMA 2. *Let* P(z) *be a function regular in* E *and normalized so that* P(0) = 1. *Let* $\Re P(z) > 0$ *in* E. *Then*

$$|2zP'(z) + 1 - P^2(z)| \le \frac{4|z|^2}{(1-|z|^2)^2} \qquad (z \in E).$$

Proof. Given P(z), we can determine a convex function $f_0(z)$ by the equation

$$1 + \frac{z f_0''(z)}{f_0'(z)} = P(z),$$

and $f_0(z)$ is unique if we require that $f_0(0) = 0$ and $f_0'(0) = 1$. Clearly,

$$\frac{f_0''(z)}{f_0'(z)} = \frac{P(z)-1}{z}, \qquad \left(\frac{f_0''(z)}{f_0'(z)}\right)' - \frac{1}{2}\left(\frac{f_0''(z)}{f_0'(z)}\right)^2 = \frac{2zP'(z) + 1 - P^2(z)}{2z^2}.$$

Lemma 2 is now a simple consequence of Lemma 1.

Proof of the theorem. Suppose $-\pi/2 < \alpha < \pi/2$ and the function $f(z) = z + a_2 z^2 + \cdots$ belongs to S_α. Let

$$P(z) = \frac{1}{\cos\alpha}\left[e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] - i\frac{\sin\alpha}{\cos\alpha}.$$

Then P(0) = 1, and condition (iii) implies that $\Re P(z) > 0$. Also,

$$1 + \frac{zf''(z)}{f'(z)} = [P(z)\cos\alpha + i\sin\alpha]e^{-i\alpha},$$

that is,

$$\frac{f''(z)}{f'(z)} = \frac{P(z)-1}{z}e^{-i\alpha}\cos\alpha.$$

Since

$$\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{e^{-i\alpha}\cos\alpha}{2z^2}[2zP'(z) - 2P(z) + 2 - e^{-i\alpha}\cos\alpha\,(P(z)-1)^2]$$

$$= \frac{e^{-i\alpha}\cos\alpha}{2z^2}[2zP'(z) + 1 - P^2(z) + ie^{-i\alpha}\sin\alpha\,(P(z)-1)^2],$$

we see that

$$\begin{aligned}|w(f, z)| &\leq \frac{\cos\alpha}{2|z|^2}\{|2z P'(z) + 1 - P^2(z)| + |\sin\alpha|\ |P(z) - 1|^2\}\\ &< \frac{\cos\alpha}{2|z|^2}\left[\frac{4|z|^2}{(1 - |z|^2)^2} + |\sin\alpha|\left(\frac{2|z|}{1 - |z|}\right)^2\right]\\ &\leq \frac{2\cos\alpha}{(1 - |z|^2)^2}[1 + 4|\sin\alpha|] \leq 2(1 - |z|^2)^{-2},\end{aligned}$$

provided α satisfies the inequality $\cos\alpha\,(1 + 4|\sin\alpha|) \leq 1$, that is, provided

$$(4\sin\alpha\cos\alpha)^2 \leq (1 - \cos\alpha)^2.$$

Hence, if either $\alpha = 0$ or

$$16\cos^3\alpha + 16\cos^2\alpha + \cos\alpha - 1 < 0,$$

then, by Nehari's test [3], $f(z)$ is univalent in E. The equation

$$16x^3 + 16x^2 + x - 1 = 0$$

has only one positive root, namely $x_0 = 0.2315\cdots$. Thus, for $0 < \cos\alpha \leq x_0$, $f(z)$ is univalent in E.

The following example is instructive (see [4]). For $\mu + 1 = |\mu + 1|\, e^{-i\alpha}$ $(-\pi/2 < \alpha < \pi/2)$, let

$$g(z) = \frac{1}{\mu}[(1 - z)^{-\mu} - 1] = z + \cdots,$$

and write

$$P(z) = e^{i\alpha}\left(1 + \frac{z g''(z)}{g'(z)}\right) = e^{i\alpha} + \frac{|\mu + 1|\, z}{1 - z}.$$

For $|z| < 1$ and $|\mu| \leq 1$,

$$\Re P(z) = |\mu + 1|\, \Re\left(\frac{1}{\mu + 1} + \frac{z}{1 - z}\right).$$

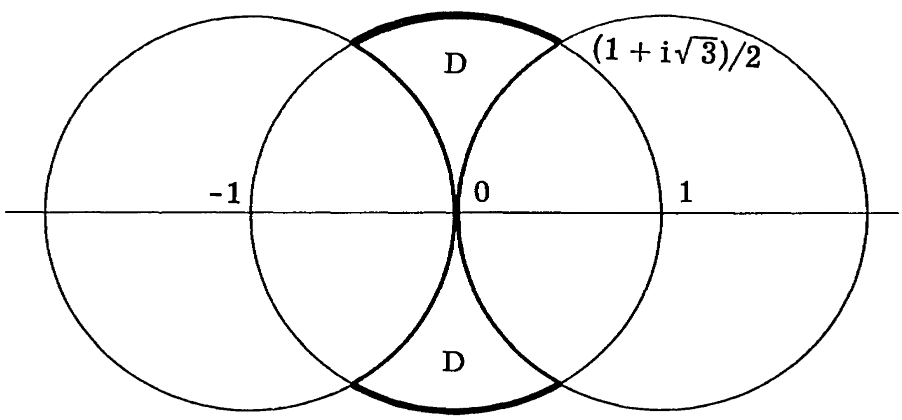

Figure 1

It follows that g(z) belongs to S_α if and only if $|\mu| \leq 1$. But in [4] it was shown that g(z) is univalent if and only if either $|\mu + 1| \leq 1$ or $|\mu - 1| \leq 1$. We conclude that g(z) belongs to S_α and is not univalent in E when μ lies in the set D defined by the inequalities $|\mu| \leq 1$, $|\mu + 1| > 1$, $|\mu - 1| > 1$ (see Figure 1). For each α for which $1/2 < \cos \alpha < 1$, we can choose μ so that $\mu + 1 = |\mu + 1| e^{-i\alpha}$ and so that μ lies in the set D. The function g(z) then belongs to S_α but is not univalent in E. Similarly, when $0 < \cos \alpha \leq 1/2$, we can choose a number $\mu + 1 = |\mu + 1| e^{-i\alpha}$ so that $|\mu + 1| \leq 1$ and $|\mu| \leq 1$, and g(z) then belongs to S_α and is univalent in E.

The question whether all functions in S_α are univalent in E remains open for the range $x_0 < \cos \alpha \leq 1/2$.

REFERENCES

1. P. L. Duren, H. S. Shapiro, and A. L. Shields, *Singular measures and domains of Smirnov type*. Duke Math. J. 33 (1966), 247-254.

2. J. Krzyż and Z. Lewandowski, *On the integral of univalent functions*. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 11 (1963), 447-448.

3. Z. Nehari, *The Schwarzian derivative and schlicht functions*. Bull. Amer. Math. Soc. 55 (1949), 545-551.

4. W. C. Royster, *On the univalence of a certain integral*. Michigan Math. J. 12 (1965), 385-387.

5. L. Špaček, *Contribution à la théorie des fonctions univalentes* (in Czech). Časop Pěst. Mat.-Fys. 62 (1933), 12-19.

University of Delaware
Newark, Delaware 19711