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INTRODUCTION

If b is an integer exceeding 1, then each positive integer n is uniquely expressible as $n = \sum_{i=0}^{\infty} \alpha(n, i) b^i$, where each $\alpha(n, i)$ is a nonnegative integer less than b. Numerous analogies exist between this representation and the representation $n = \prod_{i=1}^{\infty} p_i^{\beta(n,i)}$ given by the fundamental theorem of arithmetic. For instance, if we put $m \prec n$ if $\alpha(m, i) \leq \alpha(n, i)$ for each i, it is clear that "$\prec$" is analogous to "divides."

We say that a sequence C is b-*complete* provided that the relations $x \in C$ and $x \prec y$ imply that $y \in C$. A b-complete sequence is analogous to a sequence of integers formed by taking all multiples of the elements of a fixed sequence (sequences of this form have been studied extensively in density theory).

H. Davenport and P. Erdös [5] proved that a sequence formed by taking all multiples of the elements of a fixed sequence possesses a logarithmic density. They also proved that a sequence with positive upper logarithmic density contains a division chain, that is, an infinite subsequence in which each member divides its successor.

In giving examples of situations in which his "magnification theorem" holds, E. M. Paul [9], [10] provided another proof of the Davenport-Erdös theorems. It is implicit in the work in his dissertation [9] that a b-complete sequence possesses natural density, and that a sequence with positive upper natural density contains a chain $d_1 \prec d_2 \prec \cdots$.

In [1] we introduced a method that allowed us to extend the results of Davenport and Erdös; and in this paper we adapt this method to the study of digits. In particular, we examine the structure of chains that exist under various density conditions.

Another well-investigated type of sequence is the primitive sequence (here, no member divides another).

The following two fundamental results are due to P. Erdös [6] and F. Behrend [3], respectively: if $a_1, a_2, \cdots$ is a primitive sequence, then

1) $\sum_i (a_i \log a_i)^{-1} \leq M$, where M is an absolute constant,

2) $\sum_{a_i \leq x} 1/a_i \leq K(\log x)/\sqrt{\log \log x}$.

Our analogous results are that if $a_1, a_2, \cdots$ is a b-primitive sequence, that is, if $a_i \not\prec a_j$ whenever $i \neq j$, then

1) $\sum_i 1/a_i \leq b$,

2) $\sum_{a_i \leq x} 1 \leq Kx/\sqrt{\log x}$.

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An excellent account of much that is known regarding the multiplicative situation is given in the final chapter of Halberstam and Roth [7].

Throughout this paper, we denote by log the natural logarithm, and by $\log_b$ the logarithm to the base b.

1. REMARKS AND LEMMAS ON ASYMPTOTIC DENSITY

Let C be a sequence of positive integers, and let g be a positive, nonincreasing arithmetic function for which $\sum_{n=1}^{\infty} g(n)$ diverges. Define:

(1) $\bar{g}_k(C) = \sum \{g(c): c \in C, c \le k\} / \sum_{n=1}^{k} g(n)$,

(2) $\bar{g}^*(C) = \limsup \bar{g}_k(C)$, $\bar{g}_*(C) = \liminf \bar{g}_k(C)$,

(3) $\bar{g}(C) = \bar{g}^*(C)$ if $\bar{g}^*(C) = \bar{g}_*(C)$. If $\bar{g}_*(C) = \bar{g}^*(C)$, we say that C possesses $\bar{g}$-*density*.

We obtain natural density by setting $g(n) = 1$ for each n, logarithmic density by setting $g(n) = 1/n$. In these important cases, we shall replace the symbol $\bar{g}$ with δ and ℓ, respectively.

The first lemma is a standard elementary result on Nörlund means. For a thorough discussion, see Hardy [8, p. 58].

1.1. LEMMA. *If* g *is a positive, nonincreasing function such that* $\sum_n g(n)$ *diverges, and* C *is any sequence of positive integers, then*

$$\delta_*(C) \le \bar{g}_*(C) \le \bar{g}^*(C) \le \delta^*(C).$$

It is obvious that if $C_1, C_2, \cdots$ is a pairwise disjoint sequence of sets, each possessing $\bar{g}$-density, then the set $C = \bigcup_{i=1}^{\infty} C_i$ need not possess $\bar{g}$-density; and if C does possess $\bar{g}$-density, it need not be $\sum_i \bar{g}(C_i)$. We give a series of lemmas whose proofs depend only upon elementary analysis. If need be, a more detailed discussion may be found in Section 1 of our paper [1].

1.2. LEMMA. $\bar{g}_*(C) \ge \sum_i \bar{g}(C_i)$.

1.3. LEMMA. *If there exists a sequence of positive constants* M_i *such that* $\sum_i M_i$ *converges and* $\bar{g}_k(C_i) \le M_i$ *for each* i *and* k, *then* $\bar{g}(C)$ *exists and equals* $\sum_i \bar{g}(C_i)$.

The following is a corollary to Lemma 1.3.

1.4. LEMMA. *If* $\sum_{c \in C} [g(c)/\sum_{j=1}^{c} g(j)]$ *converges, then* $\bar{g}(C) = 0$. *In particular, if* $\sum_{c \in C} 1/c$ *converges, then* $\delta(C) = 0$; *and if* $\sum_{c \in C} 1/(c \log c)$ *converges, then* $\ell(C) = 0$.

1.5. LEMMA. *If* $E \subset C$, $\bar{g}(C) = \sum_i \bar{g}(C_i)$, *and* $\bar{g}(E \cap C_i)$ *exists for each* i, *then* $\bar{g}(E) = \sum_i \bar{g}(E \cap C_i)$.

We conclude this section with a theorem that is directly related to the remainder of the article.

1.6. THEOREM. *For* $i = 1, 2, \cdots$, *let* C_i *be the arithmetic progression* $\{a_i + kd_i: k = 0, 1, \cdots\}$, *where* a_i *is a nonnegative integer and* d_i *is a positive integer. Furthermore, suppose the sequence* $\{C_i\}$ *is pairwise disjoint. Then a necessary and sufficient condition that* $\bar{g}(C) = \sum_i 1/d_i$ *is that* $\bar{g}(A) = 0$, *where* $A = \{a_1, a_2, \cdots\}$. *In particular, if* $\sum_i 1/a_i$ *converges, then* $\delta(C) = \sum_i 1/d_i$; *if* $\sum_i 1/(a_i \log a_i)$ *converges, then* $\ell(C) = \sum_i 1/d_i$.

Proof. For each i, let $C_i' = \{a_i + kd_i: k = 1, 2, \cdots\}$. Note that for each i and k, $\delta_k(C_i') \leq 1/d_i$. Since the progressions are pairwise disjoint, $\sum_i 1/d_i \leq 1$. We apply Lemma 1.3 to conclude that $\delta(C') = \sum_i 1/d_i$, where $C' = \bigcup_i C_i'$. From Lemma 1.1 it follows that $\bar{g}(C') = \sum_i 1/d_i$. Since $C = C' \cup A$ and $A \cap C'$ is empty, it is clear that $\bar{g}(C) = \sum_i 1/d_i$ if and only if $\bar{g}(A) = 0$. The remainder of the theorem follows upon application of Lemma 1.4.

2. THE BASIC PROPERTIES OF A CERTAIN DIGITAL DECOMPOSITION

Let b, n, and $\alpha(n, i)$ be related as in the Introduction. Define

$$L(n) = \max\{i: \alpha(n, i) > 0\} \quad \text{and} \quad S(n) = \min\{i: \alpha(n, i) > 0\}.$$

Let Γ be the collection of all integer-valued arithmetic functions f for which $f(n) \geq L(n)$ for $n = 1, 2, \cdots$. For each nonnegative integer k, let

$$\Delta(k) = \{n: S(n) > k\}.$$

Observe that $\Delta(k)$ is the arithmetic progression $\{rb^{k+1}: r = 1, 2, \cdots\}$.

2.1. *Definition.* If C is a sequence of positive integers and f belongs to Γ, let A(f, C) denote the sequence of members of C that cannot be expressed as $c = c' + t$, with c' belonging to C and t belonging to $\Delta(f(c'))$; let B(f, C) be the sequence of members of C that do not belong to A(f, C). When there is no chance of confusion, these sequences will be called A and B, respectively.

2.2. LEMMA. *Let* C *be a sequence of positive integers, and let* f *belong to* Γ. *Then, if* c *belongs to* C, *either* c *belongs to* A *or* c *may be uniquely expressed as* $c = a + t$, *where* a *belongs to* A *and* t *belongs to* $\Delta(f(a))$.

Proof. Suppose c belongs to B. Then $c = c' + t$, where c' belongs to C and t belongs to $\Delta(f(c'))$. If we choose a to be the least such c', it is clear that a belongs to A. The uniqueness follows from the fact that if a_1 and a_2 are distinct members of A, then the sequences $a_1 + \Delta(f(a_1))$ and $a_2 + \Delta(f(a_2))$ are disjoint. Suppose $a_1 + t_1 = a_2 + t_2$, where t_1 belongs to $\Delta(f(a_1))$ and t_2 belongs to $\Delta(f(a_2))$. Note that either $t_1 \prec t_2$ or $t_2 \prec t_1$; if $t_1 \prec t_2$, then $a_1 = a_2 + (t_2 - t_1)$. Since $t_2 - t_1$ belongs to $\Delta(f(a_2))$, this contradicts the definition of A.

2.3. LEMMA. *For* a *in* A, $\delta[\Delta(f(a))] = b^{-[f(a)+1]}$.

2.4. THEOREM. *For each sequence* C *and each* f *in* Γ, *we have the relations*

$$\delta\left[\bigcup\{a + \Delta(f(a)): a \in A\}\right] = \sum_{a \in A} \delta[a + \Delta(f(a))] = \sum_{a \in A} b^{-[f(a)+1]}.$$

Proof. Note that for each a in A and each k, $\delta_k[a + \Delta(f(a))] \leq b^{-[f(a)+1]}$. Since the members of the collection $\{a + \Delta(f(a)) : a \in A\}$ are pairwise disjoint, $\sum_{a \in A} b^{-[f(a)+1]} \leq 1$. The result follows from Lemma 1.3.

2.5. COROLLARY. *Suppose* D *is contained in* $\bigcup \{a + \Delta(f(a)) : a \in A\}$, *and* $D_a = D \cap \{a + \Delta(f(a))\}$ *possesses* $\bar{g}$*-density* (as described in Section 1) *for each* a *in* A. *Then* $\bar{g}(D) = \sum_{a \in A} \bar{g}(D_a)$.

Proof. From Lemma 1.1 we deduce that Theorem 2.4 is valid with δ replaced by $\bar{g}$; apply Lemma 1.5.

If we pause to consider the situation where $b = 2$, $f(n) = L(n)$, and C consists of the positive integers, we note that $A = \{2^k : k = 0, 1, \cdots\}$. Theorem 2.4 implies the obvious: $\delta(B) = \sum_{k=1}^{\infty} 2^{-k} = 1$. The next few results give information on A in the general situation.

2.6. THEOREM. *Let* f *belong to* Γ, *let* C *be a sequence of positive integers, and let* A(f, C) *denote the sequence described in Definition* 2.1. *Then*

$$\sum_{a \in A} a^{-1} b^{-R(a)} \leq b,$$

where $R(n) = f(n) - L(n)$.

Proof. We know that $\sum_{a \in A} b^{-f(a)} \leq b$. To finish the proof, use the relations $f(a) = L(a) + R(a)$ and $b^{L(a)} \leq a$.

2.7. COROLLARY. *If there is a constant* K *for which* $R(n) \leq K$, *then* $\delta(A) = 0$; *similarly, if* $R(n) \leq \log_b[K \log_b n]$, *then* $\ell(A) = 0$.

Proof. If $R(n) \leq K$, then $\sum_{a \in A} 1/a \leq b^{K+1}$. If $R(n) \leq \log_b [K \log_b n]$, then $\sum_{a \in A} 1/(a \log a) \leq K'$ (K' constant). Apply Lemma 1.4.

3. SOME APPLICATIONS OF THE DECOMPOSITION

3.1. THEOREM (Paul). *If* C *is* b*-complete, then* C *possesses natural density.*

Proof. Let $f(n) = L(n)$; then $R(n)$ is identically zero. By Corollary 2.7, $\delta(A) = 0$. Since C is b-complete,

$$C \cap [a + \Delta(f(a))] = a + \Delta(f(a))$$

for each a in A. Hence we may use Corollary 2.5 to conclude that

$$\delta(C) = \delta(B) = \sum_{a \in A} b^{-[f(a)+1]}.$$

3.2. COROLLARY. *If* C *is* b*-complete, then* $\delta(A)$ *and* $\delta(B)$ *exist for each* f *in* Γ.

Proof. The existence of $\delta(B)$ is implicit in the proof of Theorem 3.1. Hence, $\delta(A) = \delta(C) - \delta(B)$.

3.3. COROLLARY. *If* g *is any function of the type discussed in Section* 1, *if* C *is* b-*complete, and if* $\bar{g}(A) = 0$, *then* $\delta(A) = 0$.

Proof. If $\bar{g}(A) = 0$, then $\delta_*(A) = 0$, by Lemma 1.1. Since $\delta(A)$ exists (by Corollary 3.2), $\delta(A) = 0$.

3.4. THEOREM. *Suppose* $\delta^*(C) > 0$, *and let* $K_1, K_2, \cdots$ *be a sequence of positive constants. Then* C *contains an infinite chain* $d_1 \prec d_2 \prec \cdots$ *such that* $b^{L(d_i)+K_i}$ *divides* $d_{i+1} - d_i$.

Proof. Let $f_i(n) = L(n) + K_i$. By Corollary 2.7, $\delta[A(f_i, D)] = 0$ for each sequence D. Since $\delta[A(f_1, C)] = 0$, there exists an a_1 in $A(f_1, C)$ for which $\delta^*(C_1) > 0$, where $C_1 = C \cap [a_1 + \Delta(f_1(a_1))]$; otherwise, Corollary 2.5 would imply that $\delta(C) = \delta(B) = 0$. Likewise, there must be an a_2 in $A(f_2, C_1)$ such that $\delta^*(C_2) > 0$, where $C_2 = C_1 \cap [a_2 + \Delta(f_2(a_2))]$. We proceed inductively, putting $d_n = a_n$.

3.5. THEOREM. *Suppose that* $\ell^*(C) > 0$ *and* $K > 0$. *Then* C *contains a chain* $d_1 \prec d_2 \prec \cdots$ *for which* $d_{i+1} - d_i$ *is divisible by* b^{s_i}, *where*

$$s_i \geq L(d_i) + \log_b[K \log_b d_i].$$

Proof. Choose $f(n) = L(n) + \log_b[K \log_b n]$. By Corollary 2.7, $\ell(A) = 0$. There must be an a_1 in A such that $\ell^*(C_1) > 0$, where $C_1 = C \cap [a_1 + \Delta(f(a_1))]$. As in Theorem 3.4, we proceed inductively, putting $a_n = d_n$.

3.6. THEOREM. *Suppose that* $\delta_*(C) > 0$, *and let* f *be any member of* Γ *for which* R(n) *is nondecreasing and* $\sum 1/nb^{R(n)}$ *diverges. Then* C *contains a chain* $d_1 \prec d_2 \prec \cdots$ *for which* $d_{i+1} - d_i$ *is divisible by* b^{s_i}, *where* $s_i \geq f(d_i)$.

Proof. Let $g(n) = 1/nb^{R(n)}$, and let $\bar{g}$ be the associated density; then $\bar{g}^*(C) \geq \delta_*(C) > 0$. Theorem 2.6 and the definition of $\bar{g}$ imply that $\bar{g}[A(f, C)] = 0$. There is a member a_1 in A for which $\bar{g}^*(C_1) > 0$, where $C_1 = C \cap [a_1 + \Delta(f(a_1))]$. We proceed as in Theorems 3.4 and 3.5.

Theorem 3.4 seems rather weak. However, the next result shows it to be best possible.

3.7. THEOREM. *Suppose* $\lim R(n) = \infty$. *Then some sequence* C, *with* $\delta^*(C) > 0$, *contains no chain* $d_1 \prec d_2 \prec \cdots$ *for which* $d_{i+1} - d_i$ *is divisible by* d^{s_i}, *where* $s_i \geq f(d_i)$.

Proof. Let K be a fixed constant exceeding 1, and let x be an integer in the interval [n, Kn]. The progression $x + \Delta(f(x))$ has density $b^{-[f(x)+1]} \leq 1/xb^{R(x)}$. The density of the totality of all such progressions as x ranges from n to Kn is less than

$$b^{-\psi(n)} \sum_{k=n}^{[Kn]} 1/k,$$

which is approximately $b^{-\psi(n)} \log K$, where $\psi(n) = \min\{R(x): x \in [n, Kn]\}$. Since $\psi(n)$ becomes large, the density tends to zero.

Let $\varepsilon > 0$ be specified, and choose the positive numbers $\rho_1, \rho_2, \cdots$ so that $2\sum_i \rho_i < \varepsilon$. We form a sequence as follows:

Choose n_i large enough so that

(1) the density of the set $\{x + t\colon x \in [n_i, Kn_i],\ t \in \Delta(f(x))\}$ is less than ρ_i;

(2) the relative density (in the interval $[n_i, Kn_i]$) of the set

$$\{x' + t'\colon x' + t' \in [n_i, Kn_i],\ x' \in [n_j, Kn_j] \text{ for some } j < i,\ t' \in \Delta(f(x'))\}$$

is less than $2\sum_{j=1}^{i-1} \rho_j$.

To construct our sequence, we take from each of the intervals $[n_i, Kn_i]$ $(i = 1, 2, \cdots)$ all elements that do not belong to the set $\{x' + t'\}$ described in (2). We note that the sequence contains no chain of the appropriate form, but that it has upper natural density exceeding $[(K - 1)/K] - \varepsilon$.

For the time being, let us assume that C is the sequence of all positive integers. If f belongs to Γ, a straightforward application of Lemma 2.2 shows that each positive integer n is uniquely expressible as $n = a_1 + \cdots + a_k$, where each a_i belongs to A and a_{i+1} belongs to $\Delta(f(a_i))$. We refer to this sum as the f-decomposition of n. Let $A_1 = A$, and let A_k be the sequence of positive integers with exactly k summands in their f-decomposition.

3.8. LEMMA. *If* f *in* Γ *is such that* $\delta(A_1) = 0$, *then* $\delta(A_k) = 0$ *for each* k.

Proof. Suppose $\delta(A_i) = 0$ for all $i \leq r$. For each a in A_1,

$$A_{r+1} \cap [a + \Delta(f(a))] = a + T,$$

where T is a subset of A_r. Hence $\delta(a + T) = 0$, and by Lemma 1.5 we conclude that $\delta(A_{r+1}) = 0$.

3.9. THEOREM. *If* f *is a member of* Γ *for which* $\sum 1/nb^{R(n)}$ *converges, then* $\delta(A_1) > 0$.

Proof. Suppose $\delta(A_1) = 0$. It follows from Lemma 3.8 that $\delta(I_r) = 1$, where I_r consists of all integers having at least r summands in their f-decomposition; also, $A(f, I_r) = A_r$. Thus, for each r,

$$1 = \delta(I_r) = \sum_{a \in A_r} b^{-[f(a)+1]} \leq \sum_{a \in A_r} 1/ab^{R(a)}.$$

As r becomes large, the least element of A_r becomes large. This leads to a contradiction by forcing the right side of the inequality to zero.

Letting $\log_b^{(k)} x$ denote the k-fold iterate of $\log_b x$, we can combine Theorems 3.6 and 3.9 to obtain the following proposition.

3.10. COROLLARY. *Let* C *be a sequence of integers for which* $\delta_*(C) > 0$. *Then, for each* r, C *contains a chain* $d_1 \prec d_2 \prec \cdots$ *for which* $d_{i+1} - d_i$ *is divisible by* b^{s_i}, *where* $s_i \geq \sum_{k=1}^{r} \log_b^{(k)} d_i$. *Such a chain need not exist if this function is replaced by*

$$\sum_{k=1}^{r} \log_b^{(k)} d_i + (1 + \varepsilon) \log_b^{(r+1)} d_i \qquad (\varepsilon > 0).$$

Readers interested in probability may recognize Corollary 3.10 as a density analogue of certain results concerning success runs in Bernoulli trials with $p = 1/b$.

Next we briefly turn our attention to the b-primitive sequences mentioned in the Introduction.

3.11. THEOREM. *If* C *is a* b-*primitive sequence, then* $\sum_{c \in C} 1/C \leq b$.

Proof. Choose $f = L$; then $A = C$. Apply Theorem 2.6.

3.12. THEOREM. *If* C *is a* b-*primitive sequence, then* $\sum_{c \leq x} 1 \leq Kx/\sqrt{\log x}$. *This result is best possible.*

Proof. It suffices to establish the equality for the case $x = b^r - 1$. We apply the pretty theorem of De Bruijn, Tengenbergen, and Kruyswijk [4] to conclude that the number of elements not exceeding $b^r - 1$ in any b-primitive sequence is at most equal to the number of integers less than b^r whose digits add exactly to the integer part of $r(b - 1)/2$. In this situation, Theorem 5 of I. Anderson [2] gives $x/\sqrt{\log x}$ as the true order of magnitude required.

We remark that for the case $b = 2$, Theorem 3.12 may be proved by an appeal to the famous lemma of Sperner on subsets of finite sets, and to a simple estimate using Stirling's formula.

FINAL REMARKS

The theorems on chains suggest the possibility of certain mappings of sequences of integers into the unit interval. Our results along these lines are so closely related to those of Paul that we merely refer the reader to his work.

We thank the referee for pointing out Anderson's theorem, which allows us to state the complete result in Theorem 3.12. We also acknowledge our debt to the vast storehouse of ideas and knowledge to be found in the papers of Professor Paul Erdös.

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