A REMARK ON FREE MODULES

K. R. Mount

Suppose A is a commutative ring with identity, and A[x] is a polynomial ring in one indeterminate with coefficients in A. Suppose F is a free module over A[x] with a basis $e_0, \cdots, e_r$. Corresponding to each element P of F, denote by $\{P\}$ the submodule of F generated by P. Set $P = \sum P_u e_u$ with

$$P_u = \sum_{v=0}^{R} p(u, v) x^v \qquad (0 \leq u \leq r),$$

and assume that $Q_u = \sum_{v=0}^{d} q(u, v) x^v$ are $r + 1$ polynomials, each of degree d, the coefficients q(u, v) being independent indeterminates. We shall denote by E(P, n; d) the matrix of the system of linear equations (in the variables q(u, v)) obtained by equating to zero the coefficients of the x^j $(0 \leq j \leq n + d)$ in the expression $\sum (-1)^u P_u Q_u$. The matrix E(P, rd; d - 1) is square. In this paper we prove the following proposition.

THEOREM. *If the determinant of* E(P, rd; d - 1) *is a unit in* A (*here* rd *denotes the maximum of the degrees of the* P_u) *and if the* A-*module*

$$[Ae_0 + \cdots + Ae_r] / \left\{ \sum p(u, rd) e_u \right\}$$

is free, then the module $F / \left\{ \sum Q_u e_u \right\}$ *is free for each* $\sum Q_u e_u$ *such that* $\sum (-1)^u Q_u P_u = 1$. *Furthermore, if* A *is an integral domain, then* $F / \left\{ \sum P_u e_u \right\}$ *is free.*

We shall suppose throughout this paper that the rings discussed are commutative and have a unit. If $P = \sum P_u e_u$ is an element of F, we shall say that P has degree d if a polynomial of maximal degree occurring among the P_u has degree d. We shall refer to the matrix E(P, n; d) as the d-th eliminate of P, and we shall suppose that the columns of E(P, n; d) are indexed by the pairs (u, v) of integers with $0 \leq u \leq r$ and $0 \leq v \leq d$, while the rows are indexed by j $(0 \leq j \leq n + d)$. In case P has degree rd, the matrix E(P, rd; d) has $(r + 1)(d + 1)$ columns and $(r + 1)d + 1$ rows; therefore, if A is a field, the dimension of the solution space of the equations $\sum P_u Q_u = 0$ (with $\deg Q_u \leq d$) is at least r.

Until we specify otherwise, we shall suppose that A is a field, and we shall denote by K an algebraically closed field of infinite degree of transcendence over A. We alter notation slightly to denote by F a free K[x]-module with basis $e_0, \cdots, e_r$. Denote by F_d the K-vector subspace of F consisting of elements of degree at most d. Denote by $G(F_d; r)$ the Grassmann space of r-dimensional subspaces of F_d. If V is a vector space, then P(V) will denote the projective space consisting of the one-dimensional subspaces of V. We shall say that an element of $P(F_d)$ has degree t if a nonzero vector in that element has degree t.

Received March 8, 1968.
This work was supported in part by N.S.F. Grant No. GP-6853.

We introduce Plüker coordinates in $G(F_d; r)$ in the usual way. Suppose $\Lambda^r{}_K F_d$ is the r-th homogeneous component of the Grassmann K-algebra of the K-space F_d, and assume $S(r, d)$ is the lexicographically ordered collection of pairs (u, v) of integers with $0 \leq u \leq r$, $0 \leq v \leq d$. If α is an increasing function from $0, \cdots, r-1$ to $S(r, d)$, set $\alpha(t) = (u(t), v(t))$, and denote by $\mathcal{E}(\alpha)$ the element

$$x^{v(0)} e_{u(0)} \wedge \cdots \wedge x^{v(r-1)} e_{u(r-1)}$$

of $\Lambda^r{}_K F_d$. The space $\Lambda^r{}_K F_d$ has a coordinate system consisting of the linear functions $X(\alpha)$ dual to the $\mathcal{E}(\alpha)$. The coordinates $X(\alpha)$ on $\Lambda^r{}_K F_d$ induce a projective coordinate system on $P(\Lambda^r{}_K F_d)$, and we shall denote by $pX(\alpha)$ the coordinate that $X(\alpha)$ determines on $P(\Lambda^r{}_K F_d)$. It is well known that if L is a point of $G(F_d; r)$ (that is, if L is a linear space of F_d and if L has a basis $B^0, \cdots, B^{r-1}$), then L may be considered as a point in $P(\Lambda^r{}_K F_d)$ with projective coordinates

$$pX(\alpha)(L) = X(\alpha)[B^0 \wedge \cdots \wedge B^{r-1}].$$

The K[x]-module F also has a Grassmann algebra $\Lambda_{K[x]} F$, the r-th homogeneous component of which we denote by $\Lambda^r F$. The module $\Lambda^r F$ is a free K[x]-module with a basis consisting of elements of the form $e_{\beta(0)} \wedge \cdots \wedge e_{\beta(r-1)} = e(\beta)$, where β is an increasing function from $0, \cdots, r-1$ to $0, \cdots, r$. Denote by β_j the increasing function from $0, \cdots, r-1$ to $0, \cdots, r$ with a range that does not contain j, and denote by $(\Lambda^r F)_{rd}$ the K-space in $\Lambda^r F$ consisting of the elements of the form $\sum P_j e(\beta_j)$, where P_j is an element of K[x] of degree at most rd. A K-basis for $(\Lambda^r F)_{rd}$ consists of the elements $x^t e(\beta_j)$. We denote by $Y(j, t)$ the linear function dual to $x^t e(\beta_j)$, and by $pY(j, t)$ the associated projective coordinate on $P((\Lambda^r F)_{rd})$.

We are now in a position to define a map Δ from $G(F_d; r)$ to $P((\Lambda^r F)_{rd})$. There exists a K-linear map I from $(\Lambda^r{}_K F)$ to $\Lambda^r F$ that carries a vector $v = v_1 \wedge \cdots \wedge v_r$ in $\Lambda^r{}_K F$ to $I(v) = v_1 \wedge \cdots \wedge v_r$ in $\Lambda^r F$, where the hook product in v is over K and the product in I(v) is over K[x]. The map I carries $\Lambda^r{}_K F_d$ linearly into the space $(\Lambda^r F)_{rd}$, and therefore it determines a linear transformation pI from $P(\Lambda^r{}_K F_d)$ to $P((\Lambda^r F)_{rd})$. The restriction of pI to $G(F_d; r)$ is the map Δ that we seek; thus Δ is a rational map from $G(F_d; r)$ to some subvariety of $P((\Lambda^r F)_{rd})$.

We need two other rather obvious descriptions of Δ. We first note that we may describe Δ by a system of linear equations. Let $J(j, s)$ be the collection of increasing functions from $0, \cdots, r-1$ to $S(r, d)$ such that $\alpha(t) = (u(t), v(t))$, $\sum u(t) = s$, and j is not in the set $u(0), \cdots, u(r-1)$. The equations for Δ are then

$$\text{(D)} \qquad pY(j, s)[\Delta(L)] = \sum pX(\alpha)[L],$$

where α ranges over $J(j, s)$. Note that if σ_j is the increasing function from $0, \cdots, r-1$ to $S(r, d)$ whose range consists of the values

$$(0, d), \cdots, (j-1, d), (j+1, d), \cdots, (r, d),$$

then $pY(j, rd)[\Delta(L)] = pX(\sigma_j)[L]$.

In order to give the second description, suppose L is a point in $G(F_d; r)$ with a basis $\sum P_{ij} e_j$ $(0 \leq i \leq r-1)$. Let M be the matrix (P_{ij}), so that M is an $r \times (r+1)$ matrix of polynomials of degree less than or equal to d. If we denote by

$\Delta_j(M)$ the determinant of the matrix derived from M by deleting the column indexed by j, then $\Delta(L)$ is the line determined by the vector $\sum \Delta_j(M)\, e(\beta_j)$.

The second map in which we shall be interested carries $P((\Lambda^r F)_{rd})$ to a subvariety of $G(F_d; r)$. Suppose that λ is an increasing function from $r, \cdots, (r+1)(d+1)$ to $S(r, d)$, and suppose L is a point in $P((\Lambda^r F)_{rd})$ determined by an element $P = \sum P_i e_i$ of F. We denote by $Z(\lambda)[P]$ the determinant of the matrix formed of the columns of $E(P, rd; d)$ indexed by the range of λ, by $c(\lambda)$ the increasing function from $0, \cdots, r-1$ to $S(r, d)$ whose range is the complement of the range of λ, and by $\varepsilon_\lambda = \pm 1$ the signature of the permutation

$$c(\lambda)(0), \cdots, c(\lambda)(r-1), \lambda(r), \cdots, \lambda(r+1)(d+1).$$

We now define a map E by the equations (D*): $pX(c(\lambda))[E(L)] = \varepsilon_\lambda Z(\lambda)[P]$. This map is obviously rational, if it is defined; moreover, it is well known [1, p. 294] that if E is defined then it carries L to the space of solutions of the system of linear equations whose matrix is the (rd, d)-eliminate. Therefore, E(L) is the K-space consisting of the solutions of the equation $\sum P_i Q_i = 0$ with $\deg Q_u \leq d$.

The criterion we seek will follow from the proof that the rational map Δ is actually a birational correspondence between $G(F_d; r)$ and $P((\Lambda^r F)_{rd})$, and that E is its inverse.

We shall say that an element P in F is generic of degree d over A if each polynomial P_u (the coefficient of e_j in P) has degree d, and if moreover, the coefficients of the polynomials P_j are algebraically independent indeterminates over A. An $r \times (r+1)$ matrix (P_{ij}) is generic of degree d over A if the P_{ij} are all of degree d and have coefficients that are algebraically independent indeterminates over A.

Suppose $\sum P_i e_i$ is an element of F and that the polynomials P_i are in $A[x]$, that they have positive degree, and that they generate the unit ideal. Suppose $\sum Q_i e_i$ and $\sum Q_i' e_i$ are generic of degree d over A, with algebraically independent indeterminates as coefficients. A simple specialization argument shows that the polynomials $\sum Q_j P_j$ and $\sum Q_j' P_j$ are relatively prime. Now suppose that $M = (P_{ij})$ is an $r \times (r+1)$ matrix that is generic of degree d over A. We claim that the polynomials $\Delta_0(M), \cdots, \Delta_r(M)$ generate the unit ideal. The proof is by induction. For $r = 1$, the assertion is clear. If $r > 1$, assume that $N = (P_{ij})$ $(0 \leq i \leq r,$ $0 \leq j \leq r+1)$ is generic of degree d. The first r columns of the determinant $\Delta_r(N)$ (or $\Delta_{r+1}(N)$) form an $(r+1) \times r$ matrix that is generic of degree d, and thus its sequence of $r \times r$ subdeterminants generates the unit ideal. If we now apply the previous remark to the expansions of $\Delta_r(N)$ and $\Delta_{r+1}(N)$ by the last row, the assertion is clear.

LEMMA. *Suppose that* $M = (P_{ij})$ *is an* $r \times (r+1)$ *matrix generic over* A *of degree* d. *If* $N = (Q_{ij})$ *is an* $r \times (r+1)$ *matrix, and if, moreover* $\lambda \Delta_j(M) = \Delta_j(N)$ $(0 \leq j \leq r)$ *for some nonzero* λ *in* K, *then the* K*-linear subspace of* F_d *spanned by the vectors* $\sum P_{ij} e_j$ $(0 \leq i \leq r-1)$ *is the same as the space spanned by the vectors* $\sum Q_{ij} e_j$ $(0 \leq i \leq r-1)$. *Further, if* $v = \sum Y_j e_j$ *is an element of* F_d *satisfying the condition* $\sum (-1)^j \Delta_j(M) Y_j = 0$, *then* v *lies in the* K*-space spanned by the* $\sum P_{ij} e_j$.

Proof. It will suffice to show that if $\sum (-1)^j Q_j \Delta_j(M) = 0$ with $\deg \left(\sum Q_j e_j\right) \leq d$, then $\sum Q_j e_j$ is a K-linear combination of the vectors $v_u = \sum P_{uj} e_j$. Denote by K(x) the field of fractions of K[x]. The vectors $v_1, \cdots, v_r$ are linearly independent over K(x), and they are solutions of the equation $\sum (-1)^j \Delta_j(M) Y_j = 0$; therefore $\sum Q_j e_j = \sum R_j^* v_j$ for some R_j^* in K(x). We can choose S and R_j $(0 \leq j \leq r)$ in K[x] that are relatively prime, such that $S\left(\sum Q_j e_j\right) = \sum R_j v_j$, in other words, such that $SQ_j = \sum R_t P_{tj}$ $(0 \leq j \leq r)$. We now apply Cramer's rule to all but the j-th equation of this system and derive the relation

$$(*) \qquad \Delta_j(M) R_k = S \det (B_{jk}),$$

where B_{jk} is an $r \times r$ matrix each element of which is either a P_{uv} or a Q_t. Since M is generic, there are polynomials C_j in K[x] satisfying the equation $\sum C_j \Delta_j(M) = 1$. If we multiply (*) by C_j and sum on j, we find that

$$R_k = S\left(\sum C_j \det (B_{jk})\right);$$

therefore S divides R_k. Since the R_t and S are relatively prime, S is an element of K. Because the matrix M is generic over A, the polynomial $\Delta_0(M)$ has degree rd and the polynomial $\det (B_{0k})$ has degree at most rd; thus the relation (*) implies that if $R_k \neq 0$, then R_k has degree zero; therefore each R_k is an element of K.

We shall now state and prove the main result.

THEOREM. *The correspondence* Δ *is a birational correspondence from* $G(F_d; r)$ *to* $P((\Lambda^r F)_{rd})$. *The rational map* E *is the inverse of* Δ. *If* L *is an element of* $P((\Lambda^r F)_{rd})$, *and if, moreover,* P *is an element of* F *of degree* rd, *then* E *is regular at* L *and* Δ *is regular at* E(L) *if and only if* $\det (E(P, rd; d-1)) \neq 0$.

Proof. We shall first show that Δ is a left inverse of E at L in $P((\Lambda^r F)_{rd})$, if L has degree rd and $\det [I(P, rd; d-1)] \neq 0$ for some P in L. We can derive the matrix $E(P, rd; d-1)$ from $E(P, rd; d)$ by deleting the $r+1$ columns indexed by (j, d) and the row indexed by $(r+1)d$. The row in $E(P, rd; d)$ indexed by $(r+1)d$ has nonzero entries only in the columns indexed by the (j, d). Suppose $P = \sum P_j e(\beta_j)$ has degree rd, and that L is the point determined by P in $P((\Lambda^r F)_{rd})$. Since P has degree rd, P_t has degree rd for some t, and therefore

$$\operatorname{rank} E(P, rd; d) = \operatorname{rank} E(P, rd; d-1) + 1 .$$

Since we have assumed $\det [E(P, rd; d-1)] \neq 0$, there are polynomials $Q_0, \cdots, Q_r$ of degree at most $d-1$ such that $\sum P_j Q_j = 1$; further, the map E is defined and regular at L. The map Δ is defined at E(L). To see this, suppose that P_j has degree rd, and denote by $\theta(j)$ the increasing function from $(r+1), \cdots, (r+1)(d+1)$ to S(r, d) whose range does not contain $(0, d), \cdots, (j-1, d), (j+1, d), \cdots, (r, d)$. Then $Z(\theta(j)) \neq 0$, and from the equations (D) defining Δ and the equations (D^*) we see that $pY(j, rd)[E(L)] = \varepsilon_{\theta(j)}\, p Z(\theta(j))[L]$.

In particular, note that if $M(L) = (Q_{ij})$ is an $r \times (r+1)$ matrix of polynomials, and if its row vectors form a basis for E(L), then the polynomial $\Delta_j(M)$ is nonzero and has degree rd. As we remarked before, E(L) is the space of solutions of the system of equations with matrix $E(P, rd; d)$; therefore $\sum (-1)^j Q_{ij} P_j = 0$. If we now apply Cramer's rule to the system of linear equations

$$\sum_{t \neq u} Q_{it} P_t = -Q_{iu} P_u,$$

we obtain the equality $\Delta_u(M) P_v = \Delta_v(M) P_u$. The P_u are relatively prime. Since $\Delta_j(M) \neq 0$, we see that $\Delta_j(M) = \Theta P_j$, and therefore $\Theta P_v = \Delta_v(M)$ for each v. The polynomials $\Delta_j(M)$ and P_j have the same degree; thus Θ is an element of K and $\Delta E(L) = L$.

To show that if $\Delta E(L) = L$, then $\det [E(P, rd; d - 1)] \neq 0$ for P in L, we note first that since E(P) is defined, rank $E(P, rd; d) = (r + 1)d + 1$, because rank $E(P, rd; d) =$ rank $E(P, rd; d - 1) + 1$. It follows immediately that $\det [E(P, rd; d - 1)] \neq 0$.

To complete the proof, we need only show that $\det [E(P, rd; d - 1)] \neq 0$ for some P that determines a point L in $P((\Lambda^r F)_{rd})$. Suppose $h = (H_{ij})$ is an $r \times (r + 1)$ matrix that is generic of degree d over A. Our lemma shows that the equation $\sum (-1)^j \Delta_j(h) Y_j = 0$ has precisely r linearly independent solutions over K; therefore, if we set $H = \sum \Delta_j(h) e(\beta_j)$, then

$$\text{rank } E(H, rd; d) = (r + 1)d + 1 ,$$

from which we conclude that $\det [E(H, rd; d - 1)] \neq 0$. This completes the proof.

Now suppose that A is an arbitrary commutative ring with a unit. If F is a free $A[x]$-module with basis $e_0, \cdots, e_r$, if $v = \sum P_j e_j$ with $P_i = \sum p(i, j) x^j$, and if the degree of v is d, then we call the A-module $Ae_0 + \cdots + Ae_r / \{v\}$ the leading coefficient module of v.

COROLLARY. *Suppose* $A[x]$ *is a polynomial ring in an indeterminate* x *over* A, *and assume* $\sum P_j e_j$ *is an element of* F *of degree* rd *and that* P_0 *is monic of degree* rd. *If* $\det [E(P, rd; d - 1)]$ *is a unit in* A, *then there exists an* $r \times (r + 1)$ *matrix* M *with entries in* $A[x]$ *such that* $\Delta_j(M) = P_j$.

Proof. Suppose first that A is a universal domain K. The map E from $P((\Lambda^r F)_{rd})$ to $G(F_d; r)$ is regular at L if P_0 is monic of degree rd and $\det [E(P, rd; d - 1)] \neq 0$ with P in L. Denote by Σ the affine open set in $P((\Lambda^r F)_{rd})$ consisting of the points L such that $Y(0, d)[L] \neq 0$ and $Z(\theta(0))[L] \neq 0$. The affine ring of Σ is then

$$K[Y(u, v)/Y(0, d)][\det (E(G, rd; d - 1))^{-1}],$$

where G is the element $\sum G_i e_i$ with $G_i = \sum_0^{rd} [Y(i, j)/Y(0, d)] x^j$. The image of Σ under E is contained in the affine open set in $G(F_d; r)$ determined by the inequality $X(\sigma_0)(L) \neq 0$, as one sees easily from the equations (D*). We may index the projective coordinates of $G(F_d; r)$ by the ranges of one-to-one functions from $0, \cdots, r - 1$ to $S(r; d)$, if we suppose that a function $X(\alpha(0), \cdots, \alpha(r - 1))$ is skew-symmetric in the sequence $\alpha(0), \cdots, \alpha(r - 1)$. With this convention, we denote by $\mu(j; u, v)$ the function with range $(1, d), \cdots, (j - 1, d), (u, v), (j + 1, d), \cdots, (r, d)$, and we introduce elements

$$B^j(y) = \sum_{(u,v)} [X(\mu(j;\, u,\, v))/X(\sigma_0)](y)x^v e_u .$$

It is well known [1, p. 313] that the linear space with basis $B^1(y), \cdots, B^r(y)$ has Plüker coordinates $[X(\gamma)/X(\sigma_0)](y)$ for each γ. If $B^j(y) = \sum Q_{jt} e_t$, set $B = [Q_{ij}(E(L))]$. The equations (D) then show that $\Delta_j(B) = P_j$ for each j. We have shown that the $B^j = \sum [X(\mu(j;\, u,\, v))/X(\sigma_0)]x^v e_u$ are elements of a free module with basis $e_0, \cdots, e_r$ over the affine ring of the set determined by the inequality $X(\sigma_0) \neq 0$, and that the identity $\Delta_j(B) = G_j$ holds. The equations (D^*) show that we may write $B^j(E(G))$ in the form

$$\sum \varepsilon_{\lambda(j;u,v)}[Z(\lambda(j;\, u,\, v))/Z(\lambda(j;\, 1,\, d))](G)x^v e_u ,$$

where $\lambda(j;\, u,\, v) = c(\mu(j;\, u,\, v))$. The equations $\Delta_j(B) = G_j$ then give an identity in the affine ring of Σ. Suppose now that A is a commutative ring with unit. If $P = \sum P_i e_i$ is an element in F satisfying the conditions of the statement of the corollary, and if we set

$$B^j(P) = \sum \varepsilon_{\lambda(j;u,v)}[Z(\lambda(j;\, u,\, v))/Z(\lambda(j;\, j,\, d))](P)x^v e_u ,$$

then since P_0 is monic, the $B^j(P)$ are elements of F and the equation $\Delta_j(B) = G_j$ specializes to $\Delta_j(B(P)) = P_j$.

Suppose now that $P_0, \cdots, P_r$ are elements of $A[x]$ where A is a commutative ring with unit, and assume that $\sum P_i Q_i = 1$ for some Q_i in $A[x]$. Denote by ϕ the homomorphism from F to $A[x]$ defined by $\phi(e_i) = P_i$. In [3 (bottom of page 162)] it is shown that if ker (ϕ) is free over $A[x]$ with generators $\sum R_{ij} e_j$ $(1 \leq i \leq r)$, then $P_i \Delta_0(R) = \varepsilon_i P_0 \Delta_i(R)$, where $\varepsilon_i = \pm 1$ and $R = (R_{ij})$. Since $\sum P_j Q_i = 1$, we deduce that $\Delta_0(R) = \left(\sum \varepsilon_i Q_i \Delta_i(R)\right)P_0$; that is, $\Delta_i(R) = HP_i$ for some H and all i. Since the ideal $P_0, \cdots, P_r$ is free, it follows from [2] that if A has connected spectrum, the $\Delta_i(R)$ generate the unit ideal, and hence H is a unit. Therefore, if A has connected spectrum, and if the kernel of ϕ is free, then there exists a matrix $R = (R_{ij})$ such that $\Delta_j(R) = P_j$.

COROLLARY. *If* A *is a commutative ring with a unit; if* $\sum P_i e_i$ *is an element of the free module* F *over* $A[x]$ *of degree* rd *such that the leading coefficient module of* $\sum P_i e_i$ *is free and such that* det $[E(P, rd;\, d - 1)]$ *is a unit, then, for each element* $Q = \sum Q_i e_i$ *in* F *satisfying the condition* $\sum (-1)^k Q_i P_i = 1$, *the module* $F/\{Q\}$ *is free. If* A *has connected spectrum, then the module* $F/\left\{\sum P_i e_i\right\}$ *is free.*

Proof. If the leading coefficient module of the element $P = \sum P_i e_i$ is free, then there exists an $(r + 1) \times (r + 1)$ matrix M with entries in A such that det (M) is a unit in A, and if $P_i = \sum p(u,\, j)x^j$, then

$$(p(0,\, rd), \cdots, p(r,\, rd))M = (1,\, 0,\, \cdots,\, 0).$$

The element $P' = \sum P_i' e_i$ with $(P_0', \cdots, P_r') = (P_0, \cdots, P_r)M$ is then in F, and P_0' is monic of degree rd. If m is a maximal ideal of A, denote by $\rho_m(P_j)$ the polynomial derived from P_j by reducing the coefficients mod m. Denote by $\rho_m(M)$ the matrix derived from M by reducing the entries modulo m, and similarly for $\rho_m E(P, rd; d-1)$. The determinant $\det[E(\rho_m P, rd; d-1)]$ is a unit in A/m. The latter statement is equivalent to the assertion that 0 is the only solution of the equations $\sum (-1)^j (\rho_m P_j) Y_j = 0$, if the Y_j are polynomials of degree at most $d - 1$. Since $\rho_m(M)$ is a unit, the equation $\sum (-1)^j (\rho_m P_j) Y_j = 0$ has only the trivial solution ($\deg Y_j \leq d - 1$), and therefore $\rho_m \det[E(P', rd; d-1)]$ is nonzero for each maximal ideal m. We may suppose that P_0 is monic of degree rd. Now we can apply the previous corollary to find an $r \times (r+1)$ matrix N with $\Delta_j(N) = P_j$. Therefore there exists an $(r+1) \times (r+1)$ matrix with determinant 1 and with first row $(Q_0, \cdots, Q_r)$. This completes the proof of the first part of the corollary. For the second part, it will suffice to show that if A has connected spectrum, if $\sum (-1)^k P_i Q_i = 1$, and if $U = F/\{Q\}$ is free, then $V = F/\{P\}$ is free. The sequence

$$0 \to \{Q\} \to F \to U \to 0$$

is exact, and U is free. We dualize by applying Hom $(\cdot, A[x])$, and we derive the exact sequence $0 \to \text{Hom}(U, A[x]) \to \text{Hom}(F, A[x]) \to A[x] \to 0$. The module Hom $(U, A[x])$ is isomorphic to the set of all $\sum R_i e_i^*$ (the e_i^* denote the basis dual to the e_i) satisfying the equation $\sum R_i Q_i = 0$. Therefore the discussion preceding the statement of the corollary shows that there exists an $r \times (r+1)$ matrix S such that $\Delta_j(S) = Q_j$. From this we see immediately that V is free.

REFERENCES

1. W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry,* Vol. I. Cambridge University Press, Cambridge, 1947.

2. K. R. Mount, *Some remarks on Fitting's invariants.* Pacific J. Math. 13 (1963), 1353-1357.

3. ———, *On polynomial rings.* Michigan Math. J. 13 (1966), 161-163.

Northwestern University
Evanston, Illinois 60201