GROUPS OF ORDER AUTOMORPHISMS OF CERTAIN HOMOGENEOUS ORDERED SETS

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1. INTRODUCTION

Call a chain (linearly ordered set) *short* if it contains a countable unbounded subset, and *homogeneous* if all convex subsets without greatest or least elements are isomorphic. The purpose of this paper is to investigate the algebraic structure of the group $S(\Omega)$ of order automorphisms of a short homogeneous chain (abbreviated SHC) Ω.

In Section 2 we show that the group structure of $S(\Omega)$ determines, up to duality, the structure of $\overline{\Omega}$ (the conditional completion of Ω) and the lattice structure of $S(\Omega)$. We give a partial solution to the problem of finding all SHC's Ω with the same group $S(\Omega)$. Our solution includes the result $S(\mathscr{R}) \not\cong S(\mathscr{Q})$.

In Section 3 we calculate the automorphism groups of large subgroups of $S(\Omega)$. Our result includes the theorem of J. T. Lloyd [5] that if Ω is conditionally complete, then every automorphism of $S(\Omega)$ comes from conjugation by an order automorphism or antiautomorphism of Ω.

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Some notation: S^{Ω} is the full group of permutations of Ω; $L(\Omega)$ (respectively, $R(\Omega)$) is the subgroup of elements of $S(\Omega)$ whose support is bounded on the right (on the left); and $N(\Omega) = R(\Omega) \cap L(\Omega)$. For unexplained terminology, see [7] and [1].

We note that not every SHC is a subset of $\mathscr{R}$ (the set of real numbers). See, for example, [6].

2. GROUP STRUCTURE AND ORDER

The following is the fundamental tool of this paper.

THEOREM 1. *If* Ω *is short, and all of its open intervals are isomorphic, then* $L(\Omega)$, $R(\Omega)$, *and* $N(\Omega)$ *are the only proper normal subgroups of* $S(\Omega)$; *also,* $N(\Omega)$ *is the only proper normal subgroup of* $L(\Omega)$ *or* $R(\Omega)$, *and* $N(\Omega)$ *is algebraically simple.*

The difficult part of this, the simplicity of $N(\Omega)$, is due to G. Higman [2] (see also [7, p. 25]). The rest of Theorem 1 is a consequence of [3, Theorem 6]. A proof also appears in [5].

Note that if Ω is isomorphic to its order dual Ω^*, then $L(\Omega) \cong R(\Omega)$ and all four of the simple factors $S(\Omega)/L(\Omega)$, $S(\Omega)/R(\Omega)$, $R(\Omega)/N(\Omega)$, and $L(\Omega)/N(\Omega)$ are isomorphic. To complete the picture, we state without proof the following theorem.

THEOREM 2. *Under the hypothesis of Theorem* 1, $N(\Omega) \not\cong R(\Omega)/N(\Omega)$.

It is not to be hoped, even if Ω is an SHC, that the algebraic structure of $S(\Omega)$ will determine Ω; for example, if Γ is the set of irrational numbers and $\mathscr{Q}$ is the

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set of rationals, then $S(\Gamma) \cong S(\mathscr{Q})$. Proof: an element of $S(\Gamma)$ determines an element of $S(\mathscr{R})$, and the restriction to $\mathscr{Q}$ of this permutation is an element of $S(\mathscr{Q})$ [3, p. 407]. However, we shall show that in a certain sense this is the only way things can go wrong (see Corollary 3 to Theorem 4). We shall also prove that up to duality the lattice structure of $S(\Omega)$ is determined by the structure of $S(\Omega)$ as an abstract group.

First we establish some more terminology. If Ω is an SHC, we set

$$U(\Omega) = \{s \in S(\Omega) \mid s \text{ can be written uniquely as the product of commuting elements } s = lr = rl \text{ with } l \in L(\Omega),\ r \in R(\Omega)\}.$$

If $s \in S(\Omega)$ has a fixed block Λ, then the element $t \in S(\Omega)$ satisfying $t \mid \Omega - \Lambda = 1$ (the identity permutation of $\Omega - \Lambda$) and $t \mid \Lambda = s \mid \Lambda$ will be called the restriction of s to Λ. We abbreviate *minimal convex fixed block* to MCFB, and we abbreviate *support* to supp. For purposes of exposition, we regard Ω as running from left to right.

THEOREM 3. *If* Ω *is an* SHC *and* $s \in S(\Omega)$, *then* $s \in U(\Omega)$ *if and only if* s *has exactly two nontrivial* MCFB's *and at most one fixed point.*

Proof, $\Rightarrow$. If s had at least three nontrivial MCFB's, then one of them would be bounded and we could write

$$s = (ln)r = r(ln) = l(nr) = (nr)l,$$

where $n \in N(\Omega)$ is the restriction of s to the bounded fixed block, and $l \in L(\Omega)$ (respectively, $r \in R(\Omega)$) is the restriction of s to the part of Ω to the left of the bounded fixed block (to the right of the bounded fixed block). Therefore $s \notin U(\Omega)$, a contradiction.

Moreover, if $s \in U(\Omega)$ had two distinct fixed points, then because it can have at most two *nontrivial* MCFB's, it would have to fix each point of some nontrivial bounded interval $[x, y]$. Then we could write

$$s = (lt^{-1})(tr) = (tr)(lt^{-1}) = lr = rl,$$

where l is the restriction of s to $(-\infty, x)$, r is the restriction of s to $(y, +\infty)$, and $t \neq 1$ is any element of $N(\Omega)$ with $\operatorname{supp} t \subset [x, y]$. Hence $s \notin U(\Omega)$, again a contradiction.

It remains to show that $s = rl \in U(\Omega)$ cannot have exactly one MCFB. Since $s \mid (-\infty, x) = l \mid (-\infty, x)$ for any x lying to the left of supp r, l must have an MCFB $\Lambda \subset \Omega$ that is unbounded to the left. Since $lrl^{-1} = r$, supp r is invariant under l and l^{-1}; hence $(\operatorname{supp} r) \cap \Lambda = \emptyset$. Consequently, Λ is an MCFB of s.

$\Leftarrow$. Let Λ_1 and Λ_2 be the two MCFB's of s, Λ_1 lying to the left of Λ_2. Let $l \in L(\Omega)$ (respectively, $r \in R(\Omega)$) be the restriction of s to Λ_1 (to Λ_2). Then $s = lr = rl$. Suppose also that $s = l'r' = r'l'$, with $l' \in L(\Omega)$ and $r' \in R(\Omega)$. As in the first part of this proof, it can be shown that $\operatorname{supp} l' \subset \operatorname{supp} l$, $\operatorname{supp} r' \subset \operatorname{supp} r$, and therefore $l = l'$, $r = r'$.

Remark. To each element lr of $U(\Omega)$ corresponds a nonextremal Dedekind cut $[D_1, D_2]$ with

$$D_1 = \text{supp } l, \qquad D_2 = \text{the closure in } \Omega \text{ of supp } r.$$

Given a nonextremal Dedekind cut of Ω, one may always find many elements of $U(\Omega)$ that correspond to it in this way.

Definition. If $lr, l'r' \in U(\Omega)$, then

$$lr \sim l'r' \Leftrightarrow l \text{ commutes with } r' \text{ and } l' \text{ commutes with } r.$$

Clearly,

$$lr \sim l'r' \Leftrightarrow lr \text{ and } l'r' \text{ correspond to the same Dedekind cut}.$$

We write $\{lr\}$ for the $\sim$-equivalence class of lr.

Definition. If $\{lr\}, \{l'r'\} \in U(\Omega)/\sim$, then

$$\{lr\} \leq \{l'r'\} \Leftrightarrow l \text{ commutes with } r'.$$

It is now easy to prove the following theorem.

THEOREM 4. *If* Ω *is an* SHC, *then* $U(\Omega)/\sim\, \cong \overline{\Omega}$, *and the representation of* $S(\Omega)$ *on* $\overline{\Omega}$ *by extension of its action on* Ω *is order-equivalent to its action on* $U(\Omega)/\sim$ *by conjugation. In particular, if* Ω_1, Ω_2 *are both* SHC*'s with* $S(\Omega_1) \cong S(\Omega_2)$, *then either* $\overline{\Omega}_1 \cong \overline{\Omega}_2$ *or* $\overline{\Omega}_1 \cong \overline{\Omega}_2^*$.

Note. The confusion between $\overline{\Omega}_2$ and $\overline{\Omega}_2^*$ arises as follows: given an abstract group that happens to be an $S(\Omega)$ for some SHC Ω, our procedure for reconstructing Ω through $U/\sim$ begins with the choice of one of the two maximal normal subgroups to play the role of $L(\Omega)$. If we choose the real $L(\Omega)$, we get the isomorphism $\overline{\Omega} \cong U/\sim$. If by accident we choose what is actually $R(\Omega)$, we get the isomorphism $U/\sim\, \cong \overline{\Omega}^*$.

The lattice structure of $S(\Omega)$ can also be recovered from $U(\Omega)/\sim$; if $s, t \in S(\Omega)$, then

$$s \leq t = \{u^s\} \leq \{u^t\} \qquad \text{for all } u \in U(\Omega).$$

Note that $U(\Omega)$ is normal in $S(\Omega)$, and that we could as well have written $\{u\}^s \leq \{u\}^t$.

COROLLARY 1. *If* Ω_1 *and* Ω_2 *are* SHC*'s with* $S(\Omega_1) \cong S(\Omega_2)$ *as groups, then either* $S(\Omega_1) \cong S(\Omega_2)$ *or* $S(\Omega_1) \cong S(\Omega_2)^*$ *as lattice-groups.*

COROLLARY 2. *If* Ω *is an* SHC *and* $\Omega \neq \overline{\Omega}$, *then* $S(\Omega) \not\cong S(\overline{\Omega})$.

Proof. $S(\overline{\Omega})$ acts transitively on $U(\overline{\Omega})/\sim$, but $S(\Omega)$ is not transitive on $U(\Omega)/\sim$ (one orbit consists of the classes of elements actually having fixed points, that is, the elements of $U(\Omega)$ corresponding to the principal cuts of Ω). In particular, $S(\mathscr{Q}) \not\cong S(\mathscr{R})$ as groups (see also [3, p. 407]).

COROLLARY 3. *If* Ω *is an* SHC, *then each* SHC Ω' *with* $S(\Omega) = S(\Omega')$ *is order-isomorphic or anti-isomorphic to some conjugacy class of* $U(\Omega)/\sim$, *and the group of order-automorphisms of each homogeneous conjugacy class is* $S(\Omega)$. *In particular, the* SHC*'s with* $S(\Omega) = S(\Omega')$ *appear as disjoint, dense subsets of* $\overline{\Omega}$.

3. THE AUTOMORPHISMS OF LARGE SUBGROUPS OF S(Ω)

Convention. If $G \leq S^{\Omega}$, with $\mathrm{Cent}_{S^{\Omega}}(G) = \langle 1 \rangle$, and if $s \in \mathrm{Norm}_{S^{\Omega}}(G)$, we shall identify s with the automorphism of G given by conjugation by s. Recall that if Ω is any set with $|\Omega| \neq 6$ and $A^{\Omega} \leq G \leq S^{\Omega}$ (A^{Ω} denotes the alternating group on Ω), then $\mathrm{Aut}(G) = \mathrm{Norm}_{S^{\Omega}}(G)$ [7, Section 4]. The next theorem is an analogue of this for groups of order automorphisms. We regard S(Ω) as a subgroup of $S(\overline{\Omega})$ in the obvious way.

THEOREM 5. *If* Ω *is an* SHC *and* $N(\Omega) \leq G \leq S(\Omega)$, *then*

$$(1)\ \mathrm{Aut}\, S(\Omega) = \mathrm{Norm}_{S^{\overline{\Omega}}}(S(\Omega)), \qquad (2)\ \mathrm{Aut}\, G = \mathrm{Norm}_{\mathrm{Aut}\, S(\Omega)}(G).$$

Proof of (1). Identify the points of $\overline{\Omega}$ with the classes in $U(\Omega)/\sim$ (see Corollary 1). Then, for $lr \in U(\Omega)$,

$$\mathrm{Stab}_{S(\Omega)}\{lr\}$$
$$= \{y \in R(\Omega) \mid y \text{ commutes with } l\} \cdot \{x \in L(\Omega) \mid x \text{ commutes with } r\}.$$

This expression for the stabilizer is invariant under automorphisms of S(Ω), and we conclude that the automorphisms of S(Ω) permute the stabilizers of points of $\overline{\Omega}$. By [7, Theorem 4.4], each automorphism is thus an element of $S^{\overline{\Omega}}$.

Remark. Holland has proved that if Ω contains a nontrivial order-complete interval and S(Ω) is transitive, then every lattice-automorphism of S(Ω) is an inner automorphism [4, Theorem 8].

The proof of part 2 of Theorem 5 depends on properties of the topology τ defined on S(Ω) by taking as a sub-base for the open neighborhoods of $s \in S(\Omega)$ the sets $B_n(s) = s\,\mathrm{Cent}_{S(\Omega)}(n)$, where $n \in N(\Omega)$. Convergence in this topology of a net $\{s_\alpha\}_{\alpha \in A} \subset S(\Omega)$ can be described in terms of Ω by

$$\lim_{\alpha \in A} s_\alpha = s \iff s_\alpha \mid \Lambda = s \mid \Lambda \text{ eventually for each bounded subset } \Lambda \text{ of } \Omega$$
$$\text{and every net } \{s_\alpha\}_{\alpha \in A} \subset S(\Omega).$$

LEMMA 1. 1) *The topology* τ *is* T_2.

2) *A sequence in* S(Ω) *converges if it is fundamental.*

3) N(Ω) *is sequentially dense in* S(Ω).

4) *Every automorphism of* S(Ω) *is a homeomorphism of* S(Ω).

5) *If* $\{m_i\}$ *and* $\{n_i\}$ *are convergent sequences in* N(Ω) *such that* $\lim m_i = \lim n_i \in S(\Omega)$ *and if* $\psi \in \mathrm{Aut}\, N(\Omega)$, *then* $\{m_i^{\psi}\}$ *and* $\{n_i^{\psi}\}$ *are convergent and* $\lim (m_i^{\psi}) = \lim (n_i^{\psi})$.

6) *If* $\{m_i\}$, $\{n_i\}$, *and* $\{l_i\}$ *are convergent sequences in* N(Ω) *and* $\psi \in \mathrm{Aut}\, N(\Omega)$, *then*

$$(\lim m_i)(\lim n_i) = \lim l_i \Rightarrow (\lim m_i^{\psi})(\lim n_i^{\psi}) = \lim l_i^{\psi}.$$

The proof is straightforward, involving only the definition of τ and the statement preceding the lemma.

LEMMA 2. Aut $N(\Omega)$ = Aut $S(\Omega)$.

Proof. Since $N(\Omega)$ is a characteristic subgroup, it suffices to show that every element $\psi \in$ Aut $N(\Omega)$ extends uniquely to a $\overline{\psi} \in$ Aut $S(\Omega)$. Define $\overline{\psi}$ as follows: for any $s \in S(\Omega)$, pick a sequence $\{n_i\}$ in $N(\Omega)$ with $\lim n_i = s$. Set $S^{\overline{\psi}} = \lim n_i^{\psi}$. Lemma 1 assures us that $\overline{\psi}$ is well-defined and $\overline{\psi} \in$ Aut $S(\Omega)$.

Proof of (2). If $\psi \in$ Aut G, then ψ is determined by its restriction to $N(\Omega)$, by Lemma 1 ($N(\Omega)^{\psi} = N(\Omega)$ because $N(\Omega)$ is normal and simple and has trivial centralizer). By Lemma 2, ψ can be extended to be an element of Aut $S(\Omega)$.

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