MINIMUM CONVEXITY OF A HOLOMORPHIC FUNCTION, II

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1. STATEMENT OF RESULTS

Let $w = f(z)$ be a nonconstant holomorphic function defined in the open unit disc D. An *arc* at $e^{i\theta}$ is a curve $A \subset D$ such that $A \cup \{e^{i\theta}\}$ is a Jordan arc. Let A be an arc at $e^{i\theta}$, parametrized by $z(t)$ $(0 \leq t < 1)$, and define a family $\mathscr{H}_A$ as follows: $H \in \mathscr{H}_A$ if and only if H is a closed half-plane in the finite w-plane W and there exists a t_0 $(0 \leq t_0 < 1)$ such that $f(z(t)) \in H$ if $t_0 \leq t < 1$. If $\mathscr{H}_A = \emptyset$, set $F_A = W$; otherwise, set $F_A = \bigcap H$, where the intersection is taken over all $H \in \mathscr{H}_A$. Note that if $f(z)$ is bounded on A, then F_A is the convex hull of the cluster set of $f(z)$ on A at $e^{i\theta}$. Our first result is the following improvement of an earlier theorem [5, Theorem 1].

THEOREM 1. *For each* $e^{i\theta}$ *there exists an arc* α *at* $e^{i\theta}$ *such that* $F_\alpha \subset F_A$ *for each arc* A *at* $e^{i\theta}$.

If $F_\alpha = \emptyset$, $f(z)$ has the limit ∞ on α at $e^{i\theta}$, and, to be sure, in a rather special way. If $F_\alpha = \{a\}$, $f(z)$ has the limit a on α at $e^{i\theta}$.

Write $f(z) = u(z) + iv(z)$, where $u(z)$ and $v(z)$ are the real and imaginary parts of $f(z)$. A real or complex-valued function $g(z)$ defined in D is said to have the (finite or infinite) *asymptotic value* a *at* $e^{i\theta}$ provided there exists an arc at $e^{i\theta}$ on which $g(z)$ has the limit a at $e^{i\theta}$. For each $e^{i\theta}$, we shall be concerned with the validity of the following proposition:

$P(\theta)$: *If* $u(z)$ *and* $v(z)$ *have the finite asymptotic values* a *and* b, *respectively, at* $e^{i\theta}$, *then* $f(z)$ *has the asymptotic value* $a + bi$ *at* $e^{i\theta}$.

An immediate consequence of Theorem 1 is that for each $e^{i\theta}$, either $f(z)$ has the asymptotic value ∞ at $e^{i\theta}$, or $P(\theta)$ holds. This result contains a theorem of Gehring and Lohwater [4]. We shall prove a considerably stronger theorem, which we proceed to state.

Let $\mathscr{L}$ be the family of straight lines L in W such that $f(z) \notin L$ if $f'(z) = 0$. Note that for each $L \in \mathscr{L}$, $f(z)$ is one-to-one on each component of the preimage $f^{-1}(L)$. Let $\mathscr{L}^*$ be the family of all half-lines L^* in W,

$$L^* = \{w + \rho e^{i\phi}: \rho \geq 0\} \qquad (w \in W,\ 0 \leq \phi < 2\pi),$$

such that $L^* \subset L$ for some $L \in \mathscr{L}$. A subset $\mathfrak{S}$ of the unit circumference C is defined as follows: $e^{i\theta} \in \mathfrak{S}$ if and only if there exists an arc at $e^{i\theta}$ that $f(z)$ maps one-to-one onto some L^* in $\mathscr{L}^*$. Clearly, on such an arc $f(z)$ has the limit ∞ at $e^{i\theta}$.

THEOREM 2. *With the possible exception of at most countably many* $e^{i\theta}$, $P(\theta)$ *holds. Any exceptional* $e^{i\theta}$ *is in* $\mathfrak{S}$.

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The modular function has the property that $P(\theta)$ fails to hold for each $e^{i\theta}$ in the countable dense set of points at which the function has the asymptotic value ∞ (for a related discussion, see Bagemihl [2]).

2. PROOFS

We first prove three independent lemmas, which will be used in the proofs of both theorems.

LEMMA 1. *Suppose that* $f(z)$ *has a finite asymptotic value* w_0 *at* $e^{i\theta}$. *Then either* $e^{i\theta} \in \mathfrak{S}$ *or* $w_0 \in F_A$ *for each arc* A *at* $e^{i\theta}$.

Proof. Suppose to the contrary that $e^{i\theta} \notin \mathfrak{S}$ and that there exists an arc A at $e^{i\theta}$ such that $w_0 \notin F_A$. We can choose A so that for some closed half-plane H, $f(A) \subset H$ and $w_0 \notin H$. Let A_0 be an arc at $e^{i\theta}$ on which $f(z)$ has the limit w_0 at $e^{i\theta}$, and such that $f(A_0) \cap H = \emptyset$. Let β be a Jordan arc lying in D that joins the initial points (that is, the endpoints in D) of A and A_0 and intersects $A \cup A_0$ only at these initial points. Let Δ be the bounded domain whose boundary is $\beta \cup A \cup A_0 \cup \{e^{i\theta}\}$. It follows from an elementary theorem on cluster sets that $f(\Delta) - H$ is dense in $W - H$ (see, for example, [3, p. 91]). Therefore there exists a $z^* \in \Delta$ such that the point $w^* = f(z^*)$ is the finite endpoint of a half-line L^* in $\mathscr{L}^*$ that intersects neither H nor the bounded set $f(\beta) \cup f(A_0)$.

Let ϕ be such that $0 \leq \phi < 2\pi$ and $L^* = \{w^* + \rho e^{i\phi}: \rho \geq 0\}$. Let Δ' be Δ minus the set of points z at which $f'(z) = 0$. Let γ be a Jordan arc such that $z^* \in \gamma \subset \Delta'$, $f(z)$ maps γ one-to-one onto a rectilinear segment Γ that is perpendicular to L^* at w^*, and such that for each $w \in \Gamma$ the half-line $L(w) = \{w + \rho e^{i\phi}: \rho \geq 0\}$ intersects neither H nor $f(\beta) \cup f(A_0)$. Then in particular $L(w)$ $(w \in \Gamma)$ does not intersect $f(\beta \cup A \cup A_0)$. For each $w \in \Gamma$, let α_w be the component of the set

$$\{z: z \in \Delta', f(z) \in L(w)\}$$

that intersects γ. Note that with the possible exception of at most countably many $w \in \Gamma$, $L(w) \in \mathscr{L}^*$, and that if $L(w) \in \mathscr{L}^*$, then α_w is an arc at $e^{i\theta}$.

For each $w \in \Gamma$, $f(z)$ maps α_w one-to-one onto a half-open segment (which may be a half-line) on $L(w)$, and $w \in f(\alpha_w)$. Since $e^{i\theta} \notin \mathfrak{S}$, we see that the length $|f(\alpha_w)|$ of $f(\alpha_w)$ is finite if $L(w) \in \mathscr{L}^*$ ($w \in \Gamma$; here and in the sequel, "length" means Euclidean length). By the standard Baire-category argument, there exist a positive number M and an open segment Γ_0 on Γ such that the set

$$\{w: w \in \Gamma_0, |f(\alpha_w)| \leq M\}$$

is dense on Γ_0. For each $w \in \Gamma_0$, let α_w^0 denote α_w minus its endpoint on γ, and set $U = \bigcup \alpha_w^0$ and $V = \bigcup f(\alpha_w^0)$, where both unions are taken over all $w \in \Gamma_0$. It is easy to see that V and U are open, and that $f(z)$ maps U one-to-one onto V. It is now clear that $|f(\alpha_w)| \leq M$ for every $w \in \Gamma_0$, and consequently that V is bounded.

We now use the well-known method of proof of the classical Gross star theorem [6, p. 292]. Choose a positive number r_0 such that $\{|z - e^{i\theta}| \leq r_0\}$ does not intersect γ, and for each r $(0 < r \leq r_0)$, set

$$c_r = U \cap \{|z - e^{i\theta}| = r\}.$$

Let $s(r)$ denote the length of $f(c_r)$ $(0 < r \leq r_0)$. Then, by Schwarz's inequality,

$$s(r)^2 = \left(\int_{c_r} |f'(z)|\, |dz| \right)^2 \leq 2\pi r \int_{c_r} |f'(z)|^2\, |dz|\,.$$

Thus, since V is bounded,

$$\frac{1}{2\pi} \int_r^{r_0} \frac{s(r)^2}{r}\, dr \leq \iint_U |f'(z)|^2\, dx\, dy < \infty \qquad (0 < r < r_0;\ z = x + iy).$$

Therefore $\liminf_{r \to 0} s(r) = 0$, since $\int_0^{r_0} \frac{s(r)^2}{r}\, dr < \infty$.

On the other hand, $s(r) \geq |\Gamma_0|$ $(0 < r \leq r_0)$, because if $L(w) \in \mathscr{L}^*$, then $\alpha_w \cap c_r \neq \emptyset$ and consequently $f(\alpha_w) \cap f(c_r) \neq \emptyset$. This is a contradiction, and the proof of Lemma 1 is complete.

LEMMA 2. *Suppose that there exist an* $L_0 \in \mathscr{L}$ *and an arc* A_0 *at* $e^{i\theta}$ *such that* $f(A_0) \cap L_0 = \emptyset$. *Let* H_0 *be the closed half-plane bounded by* L_0 *and containing* $f(A_0)$. *Let* A *be an arbitrary arc at* $e^{i\theta}$, *and let* S *be any connected subset of* L_0 *that contains* $f(A) \cap L_0$. *Then either*

(1) *there exists an arc at* $e^{i\theta}$ *that* $f(z)$ *maps one-to-one into* L_0,

or

(2) *there exists an arc* A' *at* $e^{i\theta}$ *such that* $f(A') \subset (f(A) \cap H_0) \cup S$.

Proof. Suppose that (1) does not hold. Let J be a Jordan curve such that $e^{i\theta} \in J$ and $J \subset D \cup \{e^{i\theta}\}$, and such that the interior domain Δ of J contains A_0 and A. Then each component of the set

$$\Lambda = \{z\colon z \in \Delta,\ f(z) \in L_0\}$$

is a crosscut of Δ that $f(z)$ maps one-to-one onto an open connected subset of L_0. No such component has $e^{i\theta}$ as an endpoint, because (1) does not hold.

Let U be the component of $\Delta - \Lambda$ that contains A_0. Then $f(U) \subset H_0$. Note that each arc at $e^{i\theta}$ that is contained in Δ intersects U, since otherwise a component of Λ would have $e^{i\theta}$ as an endpoint. In particular, A intersects U, and we can let A'' be an arc at $e^{i\theta}$ such that $A'' \subset A$ and the initial point of A'' is in U. If $A'' \subset U$, set $A' = A''$. Otherwise, let γ_k $(k = 1, 2, \cdots)$ be the finitely or infinitely many components of Λ that are on the boundary of U and intersect A''. Note that if there are infinitely many γ_k, the diameter of γ_k tends to 0 as $k \to \infty$.

For each k, let γ_k' be the (possibly degenerate) closed subarc of γ_k such that the endpoints of γ_k' are on A'' and $A'' \cap \gamma_k \subset \gamma_k'$. It is easy to see that there exists an arc A' at $e^{i\theta}$ such that

$$A' \subset (A'' \cap U) \cup \left(\bigcup \gamma_k' \right).$$

Note that $f(z)$ maps each γ_k' one-to-one onto a segment whose endpoints are in $f(A'') \cap L_0$, and consequently that $f(\gamma_k') \subset S$ $(k = 1, 2, \cdots)$. Thus (2) holds, and the proof of Lemma 2 is complete.

Set $\mathfrak{A} = \{e^{i\theta}: f(z)$ has a finite asymptotic value at $e^{i\theta}\}$.

LEMMA 3. *Suppose that* $e^{i\theta} \notin \mathfrak{A}$. *Let* $\{V_n\}$ *be a decreasing sequence of domains in* W *such that for each* n

(i) *either* V_n *is bounded and its boundary* ∂V_n *is a Jordan curve, or else* $\{\infty\} \cup \partial V_n$ *is a Jordan curve in the extended plane,*

(ii) $f(z) \in \partial V_n$ *if* $f'(0) = 0$,

(iii) *there exists an arc* A_n *at* $e^{i\theta}$ *such that* $f(A_n) \subset V_n$.

Then one of the following three statements holds.

(3) *For some* n *there exists an arc at* $e^{i\theta}$ *that* $f(z)$ *maps one-to-one into* ∂V_n *and on which* $f(z)$ *has the limit* ∞ *at* $e^{i\theta}$.

(4) $e^{i\theta} \in \mathfrak{S}$, *and* $f(z)$ *is bounded on some arc at* $e^{i\theta}$.

(5) *There exists an arc* α *at* $e^{i\theta}$, *described by* $z(t)$ $(0 \le t < 1)$, *and with the property that for each* n *there exists a* t_0 $(0 \le t_0 < 1)$ *such that* $f(z(t)) \in V_n$ *if* $t_0 \le t < 1$.

Proof. If for each n there is one and only one component U_n of the set

$$U_n^* = f^{-1}(V_n) \cap \{|z - e^{i\theta}| < 1/n\}$$

that contains an arc at $e^{i\theta}$ (there is at least one by hypothesis), then $U_{n+1} \subset U_n$ $(n = 1, 2, \cdots)$, and it follows readily that (5) holds.

Suppose now that (5) does not hold. Then there exist an n and distinct components U' and U" of U_n^* such that for some arcs A' and A" at $e^{i\theta}$, $A' \subset U'$ and $A'' \subset U''$. Let β be a Jordan arc lying in D, joining the initial points of A' and A", and intersecting $A' \cup A''$ only at these initial points. Let Δ be the bounded domain whose boundary is $\beta \cup A' \cup A'' \cup \{e^{i\theta}\}$. Since $U' \cap U'' = \emptyset$, there exists an arc λ at $e^{i\theta}$ such that $\lambda \subset \Delta$ and $f(\lambda) \subset \partial V_n$.

Suppose now that (3) does not hold. Then since $e^{i\theta} \notin \mathfrak{A}$, we see that V_n is bounded and that $f(z)$ assumes as a value on λ every point of ∂V_n infinitely many times. Choose $L^* \in \mathscr{L}^*$ such that if w^* denotes the finite endpoint of L^*, then $L^* \cap \overline{V}_n = \{w^*\}$ (the bar denotes closure). Let $\{z_k\}$ be a sequence of distinct points of λ such that $f(z_k) = w^*$ $(k = 1, 2, \cdots)$. Let α_k be the component of $f^{-1}(L^*)$ that contains z_k. Then $\alpha_k \cap \alpha_{k'} = \emptyset$ if $k \ne k'$. Clearly $\alpha_k \cap (A' \cup A'') = \emptyset$ $(k = 1, 2, \cdots)$. By routine arguments, at most finitely many α_k intersect β. Thus some α_k is an arc at $e^{i\theta}$, and since $e^{i\theta} \notin \mathfrak{A}$, $e^{i\theta} \in \mathfrak{S}$. Therefore, since $f(z)$ is bounded on λ, (4) holds. The proof of Lemma 3 is complete.

Proof of Theorem 1. If $e^{i\theta} \in \mathfrak{S}$, we let α be an arc at $e^{i\theta}$ that $f(z)$ maps one-to-one onto some $L^* \in \mathscr{L}^*$, and we note that $F_\alpha = \emptyset$. Suppose now that $e^{i\theta} \notin \mathfrak{S}$. If there exists an arc α at $e^{i\theta}$ on which $f(z)$ has a finite limit w_0 at $e^{i\theta}$, then $F_\alpha = \{w_0\}$, and by Lemma 1, $F_\alpha \subset F_A$ for each arc A at $e^{i\theta}$. Thus we may also suppose that $e^{i\theta} \notin \mathfrak{A}$. Define a family $\mathscr{H}$ as follows: $H \in \mathscr{H}$ if and only if H is a closed half-plane in W and there exists an arc A at $e^{i\theta}$ such that $f(A) \subset H$. We only need to consider the case where $\mathscr{H} \ne \emptyset$, for otherwise $F_A = W$ for each arc A at $e^{i\theta}$. The intersection of any family of closed sets in W is the intersection of some countable subfamily. Thus we readily see that there exists a sequence $\{H_n\}$ of closed half-planes in W such that for each n, $\partial H_n \in \mathscr{L}$ and the interior H_n^0 of

H_n contains some $H \in \mathscr{H}$, and such that $\bigcap H_n = \bigcap H$, where the last intersection is taken over all $H \in \mathscr{H}$. Set

$$V_n = \bigcap_{j=1}^{n} H_j^0 .$$

We prove by induction that for each n there exists an arc A at $e^{i\theta}$ such that $f(A) \subset V_n$. This clearly holds for $n = 1$. Suppose now that there exists an arc A at $e^{i\theta}$ such that $f(A) \subset V_{n-1}$ $(n > 1)$. Choose an $H \in \mathscr{H}$ such that $H \subset H_n^0$, and let A_0 be an arc at $e^{i\theta}$ such that $f(A_0) \subset H$. Choose $L_0 \in \mathscr{L}$ such that $L_0 \subset H_n^0 - H$, and let H_0 be the closed half-plane bounded by L_0 and containing $f(A_0)$. Then $H_0 \subset H_n^0$. We set $S = V_{n-1} \cap L_0$, and apply Lemma 2. The possibility (1) cannot occur, because $e^{i\theta} \notin \mathfrak{S} \cup \mathfrak{A}$; therefore there exists an arc A' at $e^{i\theta}$ such that

$$f(A') \subset (f(A) \cap H_0) \cup S \subset V_{n-1} \cap H_0 \subset V_n .$$

The induction is complete.

We now apply Lemma 3. Since $e^{i\theta} \notin \mathfrak{S}$ and $\partial H_n \in \mathscr{L}$ $(n = 1, 2, \cdots)$, we readily see that (3) cannot hold. Clearly, (4) cannot hold. Thus (5) holds, and it follows that $F_\alpha \subset H_n$ for each n. Consequently $F_\alpha \subset F_A$ for each arc A at $e^{i\theta}$, and the proof of Theorem 1 is complete.

Proof of Theorem 2. Define a subset $\mathfrak{Q}$ of C by the rule that $e^{i\theta} \in \mathfrak{Q}$ provided there exist finite numbers a and b that are asymptotic values (at $e^{i\theta}$) of u(z) and v(z), respectively, while $a + ib$ is not an asymptotic value of f(z) at $e^{i\theta}$. To prove Theorem 2, we prove that $\mathfrak{Q} \cap \mathfrak{A}$ and $\mathfrak{Q} - \mathfrak{A}$ are both countable subsets of $\mathfrak{S}$.

First consider $\mathfrak{Q} \cap \mathfrak{A}$. Suppose that $e^{i\theta} \in \mathfrak{Q} \cap \mathfrak{A}$, and let a, b, and w_0 be finite asymptotic values of u(z), v(z), and f(z), respectively, at $e^{i\theta}$. Then $a + bi \neq w_0$, and there exists an arc A at $e^{i\theta}$ (on which either u(z) has the limit a at $e^{i\theta}$ or v(z) has the limit b at $e^{i\theta}$) such that $w_0 \notin F_A$. Thus, by Lemma 1, $e^{i\theta} \in \mathfrak{S}$, and we have shown that $\mathfrak{Q} \cap \mathfrak{A} \subset \mathfrak{S}$. By Bagemihl's ambiguous-point theorem [1], $\mathfrak{A} \cap \mathfrak{S}$ is countable. Thus $\mathfrak{Q} \cap \mathfrak{A}$ is a countable subset of $\mathfrak{S}$.

Now consider $\mathfrak{Q} - \mathfrak{A}$. Let $\mathscr{L}_1$ be a countable family of horizontal lines in $\mathscr{L}$ whose union is dense in W, and let $\mathscr{L}_2$ be a countable family of vertical lines in $\mathscr{L}$ whose union is dense in W. Define a subset $\mathfrak{E}$ of C as follows: $e^{i\theta} \in \mathfrak{E}$ if and only if $e^{i\theta} \in \mathfrak{Q} - \mathfrak{A}$ and there exists an arc at $e^{i\theta}$ that f(z) maps one-to-one into some $L \in \mathscr{L}_1 \cup \mathscr{L}_2$. Since $\mathfrak{E} \cap \mathfrak{A} = \emptyset$, $\mathfrak{E} \subset \mathfrak{S}$; and since there exist altogether only countably many components of $f^{-1}(L)$ $(L \in \mathscr{L}_1 \cup \mathscr{L}_2)$, $\mathfrak{E}$ is countable. Thus $\mathfrak{E}$ is a countable subset of $\mathfrak{S}$, and it suffices to prove that $\mathfrak{Q} - (\mathfrak{A} \cup \mathfrak{E})$ is a countable subset of $\mathfrak{S}$. Fix $e^{i\theta} \in \mathfrak{Q} - (\mathfrak{A} \cup \mathfrak{E})$, and let a and b be finite real numbers such that u(z) and v(z) have the asymptotic values a and b, respectively, at $e^{i\theta}$. Choose a sequence $\{h_n\}$ such that $0 < h_{n+1} < h_n < 1/n$, and such that if we set

$$V_n = \{|w - (a + bi)| < h_n\},$$

then $f(z) \notin \partial V_n$ if $f'(z) = 0$.

We prove that for each n there exists an arc A^* at $e^{i\theta}$ such that $f(A^*) \subset V_n$. Fix n. Choose distinct lines L_1' and L_1'' in $\mathscr{L}_1$ and distinct lines L_2' and L_2'' in $\mathscr{L}_2$ such that if Ω_1 and Ω_2 denote the open strips whose boundaries are $L_1' \cup L_1''$ and $L_2' \cup L_2''$, respectively, then $a + bi \in \Omega_1 \cap \Omega_2$ and $\overline{\Omega}_1 \cap \overline{\Omega}_2 \subset V_n$ (the bar denotes closure). Since $u(z)$ and $v(z)$ have the asymptotic values a and b, respectively, at $e^{i\theta}$, there exist arcs A_0 and A_1 at $e^{i\theta}$ such that $f(A_0) \subset \Omega_2$ and $f(A_1) \subset \Omega_1$. Let H_2' and H_2'' be the closed half-planes bounded by L_2' and L_2'', respectively, and containing $f(A_0)$. We apply Lemma 2 with $L_0 = L_2'$, $H_0 = H_2'$, $A = A_1$, and $S = \Omega_1 \cap L_2'$. Since $e^{i\theta} \notin \mathfrak{E}$, (1) cannot hold; therefore there exists an arc A_1' at $e^{i\theta}$ such that

$$f(A_1') \subset (f(A_1) \cap H_2') \cup (\Omega_1 \cap L_2') \subset \Omega_1 \cap H_2'.$$

We now apply Lemma 2 again, this time with $L_0 = L_2''$, $H_0 = H_2''$, $A = A_1'$, and $S = \Omega_1 \cap L_2''$, and we see that there exists an arc A^* at $e^{i\theta}$ such that

$$f(A^*) \subset (f(A_1') \cap H_2'') \cup (\Omega_1 \cap L_2'') \subset \Omega_1 \cap \overline{\Omega}_2 \subset V_n.$$

Next we apply Lemma 3. Clearly, (3) cannot hold, because each V_n is bounded. If (5) holds, then $f(z(t)) \to a + bi$ as $t \to 1$, contrary to the assumption $e^{i\theta} \notin \mathfrak{A}$. Thus (4) holds. Again by Bagemihl's theorem, the set of points $e^{i\theta}$ such that (4) holds is countable. Thus $\mathfrak{Q} - (\mathfrak{A} \cup \mathfrak{E})$ is a countable subset of $\mathfrak{S}$, and the proof of Theorem 2 is complete.

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