EXTREMAL LENGTH AND p-CAPACITY

William P. Ziemer

1. INTRODUCTION

In Euclidean n-space E_n, consider two disjoint closed sets C_0 and C_1, where C_0 is assumed to contain the closure of the complement of some closed n-ball B. We follow [12] in defining the p-capacity $(1 \leq p < \infty)$ of the pair (C_0, C_1) as

$$(1) \qquad \Gamma_p(C_0, C_1) = \inf\left\{\int_{E^n} |\text{grad } u|^p \, dL_n\right\},$$

where the infimum is taken over all continuous functions u on E_n that are infinitely differentiable on $E_n - (C_0 \cup C_1)$ and assume boundary values 0 on C_0 and 1 on C_1. Serrin found this notion useful in connection with the question of removable singularities of solutions to certain partial differential equations. The case of conformal capacity is represented when p = n, and it has been fundamental in the development of a theory of quasiconformal mappings in E_n (see [7]). The importance of conformal capacity in the theory of quasiconformal mappings is partly due to an equality of Gehring [6] that relates conformal capacity to the reciprocal of the n-dimensional extremal length of all continua in E_n that intersect both C_0 and C_1. Gehring's proof is valid for a similar equality that involves p-capacity and p-dimensional extremal length, provided $p > n - 1$. It is the purpose of this paper to provide a proof for $p \geq 1$, thus answering in the affirmative question 16 of [13]. We note that the proof is elementary in the sense that it demands only a few basic facts of real function theory. Together with [4, Theorem 7], the result yields a new proof of a theorem of Wallin [14], which relates p-capacity to potential-theoretic capacity. On the other hand, our result, along with that of Wallin, establishes Fuglede's theorem for compact sets, in case k = 1.

The author wishes to thank William Gustin for a number of helpful discussions that led to improvements of some of the theorems.

2. NOTATION AND PRELIMINARIES

L_n and H^k will denote n-*dimensional Lebesgue measure* and k-*dimensional Hausdorff measure* in E_n (for properties of the latter, see [2]). If A is an L_n-measurable subset of E_n, let $\mathscr{L}^p(A)$ be the class of functions f for which $|f|^p$ is integrable, and let $\|f\|_p$ be the $\mathscr{L}^p$-norm.

2.1. A real-valued function u defined on an open subset G of E_n is called *absolutely continuous in the sense of Tonelli on* G (ACT) if it is ACT on every interval $I \subset G$ [11, p. 169]. The gradient of u (which will now be denoted by ∇u) exists L_n-almost everywhere on G; moreover, it can easily be shown that the infimum appearing in the definition (1) of p-capacity is not diminished if we extend it to the class of ACT functions that assume the specified boundary values (see [5]).

Received May 17, 1968.
This work was supported in part by a grant from the National Science Foundation.

2.2. Let χ be a set of continua in E_n. The p-*dimensional module* of χ is defined as

$$M_p(\chi) = \inf \left\{ \int_{E_n} f^p dL_n : f \wedge \chi \right\},$$

where $f \wedge \chi$ means that f is a nonnegative L_n-measurable function satisfying the condition

$$\int_\beta f\, dH^1 \geq 1$$

for every $\beta \in \chi$. The module of χ is the reciprocal of its extremal length. Observe that in this definition, f may be assumed to be lower-semicontinuous, since for every L_n-measurable function $f \in \mathscr{L}^p$ and each $\varepsilon > 0$, there is a lower-semicontinuous function $g \geq f$ such that

$$\int g^p dL_n < \int f^p dL_n + \varepsilon.$$

We shall not need the following result in its full strength, but we enter it here for the sake of completeness. For $p \geq 2$, the proof was given in [15], and in a private communication to the author, M. Ohtsuka presented a proof for $1 < p < 2$. The proof below represents a consolidation of ideas.

2.3. LEMMA. *Let* $\chi = \bigcup_{i=1}^\infty \chi_i$, *where* $\chi_1 \subset \chi_2 \subset \cdots$ *are sets of continua in* E_n. *If* $1 < p < \infty$, *then*

$$M_p(\chi) = \lim_{i \to \infty} M_p(\chi_i).$$

Proof. For each i, there is a measurable function f_i such that

$$f_i \wedge \chi_i', \qquad \chi_i' \subset \chi_i, \qquad M_p(\chi_i') = M_p(\chi_i), \qquad \|f_i\|_p^p = M_p(\chi_i)$$

(see [4, Section 2]). If $i < j$, then it can be arranged so that $\chi_i' \subset \chi_j'$ and consequently $2^{-1} \cdot (f_i + f_j) \wedge \chi_i'$. Hence, $M_p(\chi_i) \leq 2^{-p} \|f_i + f_j\|_p^p$; in other words,

$$2^{p'} [M_p(\chi_i)]^{1/(p-1)} \leq \|f_i + f_j\|_p^{p'}, \tag{2}$$

where $p' = p/(p-1)$. Let $M_p^* = \lim_{i\to\infty} M_p(\chi_i)$, and assume, without loss of generality, that $M_p^* < \infty$. Then, if $p \geq 2$, it follows from (2) and Clarkson's inequality [1] that

$$2^p M_p(\chi_i) + \|f_i - f_j\|_p^p \leq \|f_i + f_j\|_p^p + \|f_i - f_j\|_p^p \leq 2^{p-1} [\|f_i\|_p^p + \|f_j\|_p^p] \leq 2^p M_p^*,$$

Similarly, for $1 < p < 2$,

$$\begin{aligned} 2^{p'} [M_p(\chi_i)]^{1/(p-1)} + \|f_i - f_j\|_p^{p'} &\leq \|f_i + f_j\|_p^{p'} + \|f_i - f_j\|_p^{p'} \\ &\leq 2^{p'} [2^{-1} \|f_i\|_p^p + 2^{-1} \|f_j\|_p^p]^{1/(p-1)} \\ &\leq [2^p M_p^*]^{1/(p-1)} \leq 2^{p'} M_p^{*1/(p-1)}. \end{aligned}$$

Thus, for $1 < p < \infty$, $\{f_i\}$ is a fundamental sequence in $\mathscr{L}^p$; therefore, there exists an f such that $\|f_i - f\|_p \to 0$. By appealing to [4, Theorem 3], we conclude that for an appropriate subsequence there exists a subset $\chi^* \subset \chi' = \bigcup_{i=1}^{\infty} \chi_i'$ such that

$$M_p(\chi^*) = M_p(\chi') = M_p(\chi) \quad \text{and} \quad \lim_{i\to\infty} \int_\beta |f_i - f|\, dH^1 = 0 \quad \text{for each } \beta \in \chi^* .$$

But every β in χ^* is in some χ_j', and since $f_i \wedge \chi_j'$ for $i \geq j$, it follows that $f \wedge \chi^*$. Hence,

$$M_p(\chi) = M_p(\chi^*) \leq \|f\|_p^p = \lim_{i\to\infty} \|f_i\|_p^p = \lim_{i\to\infty} M_p(\chi_i);$$

this concludes the proof.

2.4. COROLLARY. *If* $1 \leq p < \infty$ *and* $M_p(\chi_i) = 0$ *for* $i = 1, 2, \cdots$, *then* $M_p(\chi) = 0$.

Proof. For each i, there exists a measurable function f_i such that $f_i \wedge \chi_i$ and $\|f_i\| < i^{-1}$. Again by [4, Theorem 3], there exists a subset $\chi^* \subset \chi$ such that

$$M_1(\chi^*) = M_1(\chi) \quad \text{and} \quad \lim_{i\to\infty} \int_\beta f_i\, dH^1 = 0 \quad \text{for each } \beta \in \chi^* .$$

But each β in χ^* is in some χ_j, and since $f_i \wedge \chi_j$ for $i \geq j$, it is clear that χ^* must be empty, and therefore that $M_1(\chi) = 0$.

3. EQUIVALENCE OF p-CAPACITY AND p-MODULUS

We recall that if $H^1(\beta) < \infty$ for some continuum β in E_n, then β is locally connected and therefore arcwise connected (if β were not locally connected, there would be a point $x \in \beta$ with the property that for all sufficiently small balls B centered at x, $(\partial B) \cap \beta$ would contain infinitely many points, and thus, by a general inequality concerning Hausdorff measures [3, Theorem 3.2], it would follow that $H^1(\beta) = \infty$).

3.1. LEMMA. *If* χ *is the set of all continua in* E_n *that intersect both* C_0 *and* C_1, *then* $\Gamma_p(C_0, C_1) \geq M_p(\chi)$ $(p \geq 1)$.

Proof. It suffices to prove, for each infinitely differentiable function u on $E_n - (C_0 \cup C_1)$ that is admitted into the class of functions over which the infimum is taken in (1), that

$$\int_\beta |\nabla u|\, dH^1 \geq 1,$$

where β belongs to χ and is contained in B. We may assume that $H^1(\beta) < \infty$, since the set of $\beta \in \chi$ for which $H^1(\beta) = \infty$ has p-dimensional module 0. Thus, β can be taken to be arcwise connected, and consequently there is an arc

$$\beta^* \subset \beta \cap [E_n - (C_0 \cup C_1)]$$

joining two points x_0 and x_1 $(x_i \in C_i)$. Since $H^1(\beta^*) < \infty$, there exists an arc-length parametrization of β^*, say γ: $[0, a] \to \beta^*$, such that $\gamma(0) = x_0$, $\gamma(a) = x_1$, and

$a = H^1(\beta^*)$ [9, p. 259]. Now γ has Lipschitz constant 1, and $|\gamma'| = 1$ a.e. on $[0, a]$. This implies that $u \circ \gamma$ is a Lipschitz function, and therefore

$$1 \leq \int_0^a |(u \circ \gamma)'| \, dL_1 \leq \int_0^a |\nabla u \circ \gamma| \cdot |\gamma'| \, dL_1 = \int_{\beta^*} |\nabla u| \, dH^1 = \int_\beta |\nabla u| \, dH^1 .$$

3.2. The remainder of this section will be devoted to the proof of the inequality opposite to that in Lemma 3.1. To establish this inequality, it is sufficient to show that

$$\Gamma_p(C_0, C_1) \leq \int_{E_n} f^p \, dL_n ,$$

where f is a function such that $f \wedge \chi$. Observe that the family χ can be assumed to be contained in the closed ball B, the closure of whose complement is contained in C_0. Referring to Section 2.2, we see that we can assume f to be lower-semicontinuout. Moreover, f can also be assumed to be bounded away from zero on int B. To see this, let

$$f_i(x) = \begin{cases} f(x) & \text{if } f(x) > i^{-1} \text{ and } x \in \text{int } B, \\ i^{-1} & \text{if } f(x) \leq i^{-1} \text{ and } x \in \text{int } B, \\ 0 & \text{otherwise}, \end{cases}$$

for each positive integer i. Then f_i is lower-semicontinuous, $\int_\beta f_i \, dH^1 \geq 1$ for every $\beta \in \chi$, and $\|f_i - f\|_p \to 0$ over B, since B is bounded.

3.3. LEMMA. *Suppose* $\{\beta_i\}$ *is a sequence of continua in* B *such that for some* $M > 0$, $H^1(\beta_i) < M$ *for all* i. *Further, suppose that there are points* $x_i, y_i \in \beta_i$ *such that* $x_i \to x$, $y_i \to y$. *Then, there is a continuum* β *in* B *containing* x *and* y *such that*

$$\liminf_{i \to \infty} \int_{\beta_i} f \, dH^1 \geq \int_\beta f \, dH^1$$

for every lower-semicontinuous function $f \colon B \to E_1$.

Proof. Using again the fact that a continuum of finite one-dimensional Hausdorff measure is arcwise connected, we see that there is an arc $\beta_i^* \subset \beta_i$ whose end points are x_i and y_i, since $H^1(\beta_i) < M < \infty$. Thus, there is an arc-length parametrization of β_i^*, say $\gamma_i \colon [0, a_i] \to \beta_i^*$, where $\gamma_i(0) = x_i$, $\gamma_i(a_i) = y_i$, and $a_i = H^1(\beta_i^*)$. The function γ_i is Lipschitzian with Lipschitz constant 1, and $|\gamma_i'| = 1$ a.e. on $[0, a_i]$. Since γ_i is Lipschitzian, it can be extended to $[0, M]$ with the same Lipschitz constant [10, p. 341]. Hence, the sequence $\{\gamma_i\}$ is equicontinuous on $[0, M]$ and uniformly bounded, since the ranges of all γ_i touch a fixed compact set. By Ascoli's theorem, there exist a subsequence (which will still be denoted by $\{\gamma_i\}$) and a map $\gamma \colon [0, M] \to E_n$ such that $\{\gamma_i\}$ converges to γ uniformly on $[0, M]$. Note that γ has Lipschitz constant 1. By passing to another subsequence, we may assume that $a_i \to a \in [0, M]$. In view of the fact that $|\gamma_i(a_i) - \gamma_i(a)| \leq |a_i - a| \to 0$, it is clear that $\gamma_i(a) \to \gamma(a) = y$. Let $\beta = \gamma([0, a])$, and observe that β contains x and y. For every $\varepsilon > 0$, $a_i > a - \varepsilon$ for large i. Therefore, Fatou's lemma, the lower-semicontinuity of f, and the facts that $|\gamma_i'| = 1$ a.e. and $|\gamma'| \leq 1$ a.e. yield the relations

$$\liminf_{i\to\infty} \int_{\beta_i} f\, dH^1 \geq \liminf_{i\to\infty} \int_{\beta_i^*} f\, dH^1 = \liminf_{i\to\infty} \int_0^{a_i} f\circ\gamma_i\, |\gamma_i'|\, dL_1$$

$$\geq \liminf_{i\to\infty} \int_0^{a-\varepsilon} f\circ\gamma_i\, dL_1 \geq \int_0^{a-\varepsilon} f\circ\gamma\, dL_1 \geq \int_0^{a-\varepsilon} f\circ\gamma\, |\gamma'|\, dL_1 .$$

Since ε is arbitrary, we conclude that

$$\liminf_{i\to\infty} \int_{\beta_i} f\, dH^1 \geq \int_0^{a} f\circ\gamma\, |\gamma'|\, dL_1 = \int_{\beta} f\, dH^1 .$$

3.4. The function f introduced in Section 3.2 will be modified once again. Assuming that f is bounded away from zero on B, that f is lower-semicontinuous, and that $f \wedge \chi$, we define for every positive integer k the function

$$f_k(x) = \begin{cases} f(x) & \text{if } f(x) \leq k, \\ k & \text{if } f(x) > k, \end{cases}$$

which is still lower-semicontinuous. For every $x \in B$, define

$$u_k(x) = \inf \left\{ \int_{\beta} f_k\, dH^1 \right\},$$

where the infimum is taken over all continua β in B that intersect both $\{x\}$ and C_0. We shall now show that the infimum is actually attained.

Suppose $\{\beta_i\}$ is a minimizing sequence in the definition of $u_k(x)$. Then $\{\beta_i\}$ contains x and a point $y_i \in C_0 \cap B$. Since $C_0 \cap B$ is compact, we may assume that $y_i \to y \in C_0 \cap B$. By appealing to Lemma 3.3, we find that there exists a continuum β containing x and y such that

$$u_k(x) = \lim_{i\to\infty} \int_{\beta_i} f_k\, dH^1 \geq \int_{\beta} f_k\, dH^1 \geq u_k(x).$$

3.5. We recall the notion of approximate continuity and the fact that an L_n-measurable function $f\colon E_n \to E_1$ is approximately continuous in each variable at L_n-almost every point in E_n [11, pp. 132, 298]. Moreover, the derivative of the indefinite integral of a bounded, measurable function g of a real variable exists and is equal to g at points where g is approximately continuous.

Applying these remarks to the function f_k, we conclude that at L_n-almost every point $x \in E_n$, there is a countable, dense set $\Lambda_k(x)$ of vectors, each issuing from x and having unit length, such that

$$\lim_{t\to 0} t^{-1} \int_{t\lambda} f_k(y)\, dH^1(y) = f_k(x) \tag{3}$$

for every $\lambda \in \Lambda_k(x)$.

3.6. LEMMA. *The function* u_k *has Lipschitz constant* k *and satisfies the inequality* $|\nabla u_k(x)| \leq f_k(x)$ *for almost every* $x \in B$ *in the sense of* L_n.

Proof. Choose x_1 and x_2 in B, and let β_1 be a continuum that minimizes the infimum in the definition of $u_k(x_1)$ (see Section 3.4). The line segment λ that joins x_1 to x_2 is in B. Hence,

$$u_k(x_2) \leq \int_{\beta_1} f_k dH^1 + \int_\lambda f_k dH^1 \leq u_k(x_1) + kH^1(\lambda).$$

To establish the second part of the lemma, consider a point $x \in B$ at which $\Lambda_k(x)$ is defined (see Section 3.5) and at which u_k is differentiable [10, p. 336]. It will be sufficient to show that

$$Du_k(x, v) \leq f_k(x), \tag{4}$$

where $Du_k(x, v)$ denotes the directional derivative of u_k at x in the direction of the unit vector v.

Choose $\varepsilon > 0$, and select $\lambda \in \Lambda_k(x)$ with the property that if α_t denotes the line segment joining the end points of tv and $t\lambda$, then $H^1(\alpha_t) < \varepsilon t$, for $t > 0$. Then

$$u_k(x + tv) \leq u_k(x) + \int_{t\lambda} f_k dH^1 + \int_{\alpha_t} f_k dH^1 < u_k(x) + \int_{t\lambda} f_k\, dH^1 + k\varepsilon t.$$

In view of (3), the above inequalities imply (4), since ε is arbitrary; the proof of the lemma is thus concluded.

Obviously, the function u_k vanishes on C_0. Since u_k is continuous on E_n, the number

$$m_k = \min \{u_k(x): x \in C_1\}$$

is well-defined.

3.7. LEMMA. $\liminf_{k\to\infty} m_k \geq 1$.

Proof. Let $x_k \in C_1$ be such that $u_k(x_k) = m_k$. Recall from Section 3.4 that there is a continuum $\beta_k \subset B$ that intersects both $\{x_k\}$ and C_0 such that

$$u_k(x_k) = \int_{\beta_k} f_k dH^1.$$

If we assume that $\liminf_{k\to\infty} m_k < 1$, then some subsequence would satisfy the inequality

$$\int_{\beta_k} f_k dH^1 < 1.$$

Since f_k is assumed to be bounded away from zero by a number $c > 0$, the last inequality implies that $H^1(\beta_k) < c^{-1}$ for infinitely many k. Now, an application of Lemma 3.3 produces a continuum $\beta \subset B$ that intersects both C_0 and C_1. Hence, $\int_\beta f\, dH^1 \geq 1$, where f is as in Section 3.2. For $\varepsilon > 0$, choose m so that

$$\int_\beta f_m \, dH^1 > \int_\beta f \, dH^1 - \varepsilon > 1 - \varepsilon .$$

For $k \geq m$, we obtain the relation

$$m_k = \int_{\beta_k} f_k \, dH^1 \geq \int_{\beta_k} f_m \, dH^1 ,$$

and Lemma 3.3 now implies that

$$(5) \quad \liminf_{k \to \infty} m_k = \liminf_{k \to \infty} \int_{\beta_k} f_k \, dH^1 \geq \liminf_{k \to \infty} \int_{\beta_k} f_m \, dH^1 \geq \int_\beta f_m dH^1 > 1 - \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, (5) contradicts the assumption $\liminf_{k \to \infty} m_k < 1$.

3.8. THEOREM. *If* $p \geq 1$, *then* $\Gamma_p(C_0, C_1) = M_p(\chi)$.

Proof. By Section 3.2 it suffices to show that

$$\Gamma_p(C_0, C_1) \leq \int_{E_n} f^p \, dL_n .$$

To this end, let u_k^* be the truncation of u_k at level m_k, and observe that $m_k^{-1} \cdot u_k^*$ (as discussed in Section 2.1) is an admissible function for $\Gamma_p(C_0, C_1)$. From Lemma 3.6 we see easily that $|\nabla u_k^*| \leq f_k$ a.e. in E_n. Finally, Lemma 3.7 leads to the inequalities

$$\Gamma_p(C_0, C_1) \leq \liminf_{k \to \infty} m_k^{-p} \int_{E_n} |\nabla u_k^*|^p dL_n \leq \liminf_{k \to \infty} m_k^{-p} \int_{E_n} f_k^p dL_n \leq \int_{E_n} f^p \, dL_n .$$

4. POTENTIAL-THEORETIC CAPACITY AND p-CAPACITY

In this section we show that the results of Wallin [14] and Fuglede [4, Theorem 7] are equivalent, for compact sets and $k = 1$.

4.1. *Definition.* The *p-capacity* of a compact set $A \subset E_n$ is defined to be

$$\Gamma_p(A) = \inf \left\{ \int_{E_n} |\nabla u|^p \, dL_n \right\},$$

where the infimum is taken over all functions $u \in C^\infty$ that have compact support and are identically equal to 1 on A. If $p \geq n$, then the support of each function u is required to lie in some fixed sphere.

For every $r > 0$, let $\Gamma_p^r(A)$ be as in Definition 4.1, except that we require the supports of the functions u to be contained in an r-neighborhood of A.

4.2. LEMMA. *If* $1 \leq p < n$ *and* $r > 0$, *then* $\Gamma_p^r(A) = 0$ *if and only if* $\Gamma_p(A) = 0$.

Proof. Only the "if" direction requires proof, and we may assume that $1 \leq p < n$ (otherwise, there would be little to verify).

Let $\{u_i\}$ be a sequence of C^∞-functions with compact supports, identically 1 on A, and suppose that $\|\nabla u_i\|_p \to 0$ as $i \to \infty$. Let ϕ be a C^∞-function that equals 1 on A and whose support is contained in an r-neighborhood of A. Then $\phi \cdot u_i$ is admissible in the definition of $\Gamma_p^r(A)$, and

$$\|\nabla(\phi u_i)\|_p \leq \|\nabla\phi \cdot u_i\|_p + \|\phi \cdot \nabla u_i\|_p .$$

Consider the Sobolev inequality

$$\|u\|_r \leq \text{const. } \|\nabla u\|_p \qquad (r = np/(n - p)).$$

Applying this inequality to the functions u_i, we deduce for a subsequence of the u_i that $u_i \to 0$ a.e. Since the u_i can be assumed to be bounded above by 1, we conclude that $\|\nabla\phi \cdot u_i\|_p \to 0$ as $i \to \infty$. Obviously, $\|\phi \cdot \nabla u_i\|_p \to 0$, and therefore $\|\nabla(\phi u_i)\|_p \to 0$.

Let $\chi(A)$ denote the set of all continua that intersect A.

4.3. THEOREM. *If* $1 \leq p < n$, *then* $M_p[\chi(A)] = 0$ *if and only if* $\Gamma_p(A) = 0$.

Proof. Let $[A]_r$ denote the r-neighborhood of A, and let $\chi_r(A)$ be the set of all continua that join A to $E_n - [A]_r$.

If $M_p[\chi(A)] = 0$, then clearly $M_p[\chi_r(A)] = 0$ for all $r > 0$. But Theorem 3.8 implies that $\Gamma_p^r(A) = 0$, which in view of Lemma 4.2 implies that $\Gamma_p(A) = 0$.

Conversely, if $\Gamma_p(A) = 0$, then (by Lemma 4.2) $\Gamma_p^r(A) = 0$ for every $r > 0$. Now Theorem 3.8 implies that $M_p[\chi_r(A)] = 0$, and we conclude that

$$\chi(A) = \bigcup_{r>0} \chi_r(A).$$

Consequently, Corollary 2.4 shows that $M_p[\chi(A)] = 0$.

The equivalence of the results of Wallin [14] and Fuglede [4, Theorem 7] for compact sets and $k = 1$ follows immediately from Theorem 4.3; for Wallin relates potential-theoretic capacity to Γ_p, whereas Fuglede relates it to $M_p[\chi(A)]$.

We conclude by stating that if Σ denotes the class of closed sets separating C_0 from C_1 in B and if $1 < p < \infty$, then

$$(\Gamma_p)^{1/(p-1)} = M_{p'}(\Sigma) \qquad (p' = p/(p - 1)). \tag{6}$$

Relation (6) extends the results in [15]. Its proof proceeds along the same lines as the proof in [15], once an extremal for Γ_p has been obtained. The methods of [7] provide an extremal in case $p > n - 1$. However, by using the methods of [8, Chapter 3], we can show that the value of Γ_p does not change if we enlarge the class of competing functions in the definition of Γ_p $(p \geq 1)$. An extremal exists in this larger class, and this along with the proof in [15] establishes relation (6).

REFERENCES

1. J. A. Clarkson, *Uniformly convex spaces.* Trans. Amer. Math. Soc. 40 (1936), 396-414.

2. H. Federer, *The* (ϕ, k) *rectifiable subsets of* n *space.* Trans. Amer. Math. Soc. 62 (1947), 114-192.

3. ——— , *Some integralgeometric theorems.* Trans. Amer. Math. Soc. 77 (1954), 238-261.

4. B. Fuglede, *Extremal length and functional completion.* Acta Math. 98 (1957), 171-219.

5. F. W. Gehring, *Symmetrization of rings in space.* Trans. Amer. Math. Soc. 101 (1961), 499-519.

6. ——— , *Extremal length definitions for the conformal capacity of rings in space.* Michigan Math. J. 9 (1962), 137-150.

7. ——— , *Rings and quasiconformal mappings in space.* Trans. Amer. Math. Soc. 103 (1962), 353-393.

8. C. B. Morrey, Jr., *Multiple integrals in the calculus of variations.* Grundlehren math. Wissensch. Vol. 130. Springer-Verlag, New York, 1966.

9. T. Rado, *Length and area.* Amer. Math. Soc. Colloquium Publications Vol. 30. Amer. Math. Soc., New York, 1948.

10. T. Rado and P. Reichelderfer, *Continuous transformations in analysis.* Grundlehren math. Wissensch. Vol. 75. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.

11. S. Saks, *Theory of the integral.* Warsaw, 1937.

12. J. Serrin, *Removable singularities of solutions of elliptic equations.* Arch. Rational Mech. Anal. 17 (1964), 67-78.

13. G. Springer, Editor, *Proceedings of the symposium on quasiconformal mapping at Indiana University.* J. Math. Mech. 16 (1967), 1061-1067.

14. H. Wallin, *A connection between* α*-capacity and* L^p*-classes of differentiable functions.* Ark. Mat. 5 (1963-65), 331-341.

15. W. P. Ziemer, *Extremal length and conformal capacity.* Trans. Amer. Math. Soc. 126 (1967), 460-473.

Indiana University
Bloomington, Indiana 47401