A COEFFICIENT PROBLEM FOR A CLASS OF UNIVALENT FUNCTIONS

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1. INTRODUCTION

Let $E_r = \{z: |z| < r\}$, let $E_1 = E$, and let

$$S_r = \{f: f \in S \text{ and } f(E) \supset E_r\},$$

where S denotes the collection of functions $f(z) = z + a_2 z^2 + \cdots$ that are regular and univalent in E. Let S_r^* consist of the functions $f \in S_r$ for which f(E) is starlike, and write $S_{1/4}^* = S^*$. Consider the following extremal problems.

Problem 1. Find $\max |a_2|$ for $f(z) = z + a_2 z^2 + \cdots \in S_r$.

Problem 2. Find $\max |a_2|$ for $f(z) = z + a_2 z^2 + \cdots \in S_r^*$.

It is clear that for $r \leq 1/4$ the Koebe function solves both problems, and that for $r = 1$ the function $f(z) = z$ solves both problems.

In this paper we solve Problem 2, and we make a conjecture concerning Problem 1. We also conjecture that for the class σ of biunivalent functions (see [4]), the coefficient a_2 satisfies the sharp inequality $|a_2| \leq 4/3$. We give an example of a function $f \in \sigma$ for which $|a_2| = 4/3$.

For $1/4 < r < 1$, an extremal domain for Problem 2 consists of the entire complex plane minus a set $\{z: |z| \geq r, \pi - \psi \leq \arg z \leq \pi + \psi\}$ $(0 < \psi < \pi)$. To be more specific, if we choose ϕ so that

$$(1) \qquad r = [(1+\cos\phi)^{1+\cos\phi}(1-\cos\phi)^{1-\cos\phi}]^{-1} \qquad (0 < \phi < \pi/2),$$

then $|a_2| = 2\cos^2\phi$ for the extremal function $f(z) = z + a_2 z^2 + \cdots$; and if we take $a_2 > 0$, the extremal domain is as described above, with $\psi = \pi(1 - \cos\phi)$.

It is interesting to note the relation between this solution and the solution to another extremal problem. Let $M > 1$, and consider the functions $g(z) = z + b_2 z^2 + \cdots$ in S with $|g(z)| < M$ $(z \in E)$. The function $g \in S$ determined by the differential equation

$$\frac{z g'(z)}{g(z)} = G(z) = \left(1 + \frac{4\left(1 - \frac{1}{M}\right) z}{(1-z)^2}\right)^{-1/2} \qquad (G(0) = 1)$$

maximizes $|b_2|$ in this class of functions [5, p. 244, Exercise 4]. If we take $\cos^2\phi = 1 - \frac{1}{M}$ and r has the value in (1), then the function $f \in S$ satisfying the equation

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$$\frac{zf'(z)}{f(z)} = \frac{1}{G(z)} = \left(1 + \frac{4z\cos^2\phi}{(1-z)^2}\right)^{1/2} \tag{2}$$

solves Problem 2.

2. PROPERTIES OF A PARTICULAR STARLIKE MAPPING

We now show that the function f, given by (2) with $0 < \phi < \pi/2$, has the properties claimed for the solution of Problem 2.

Consider

$$F(z) = \frac{zf'(z)}{f(z)} = \left(1 + \frac{4z\cos^2\phi}{(1-z)^2}\right)^{1/2} \qquad (F(0) = 1,\ 0 < \phi < \pi/2).$$

From the properties of the Koebe function it follows that the function $F^2(z)$ maps the disk E onto the plane minus the slit $w \leq 1 - \cos^2\phi$. Hence $F(z)$ maps the disk onto the half-plane $\Re w > 0$ minus the slit $0 \leq w \leq \sin\phi$. Therefore $f \in S^*$.

Now let $u + iv = F(e^{i\theta})$ $(2\pi > \theta > 0)$. Again it follows from the properties of the Koebe function that $u = 0$ and $v > 0$ when $\pi - 2\phi > \theta > 0$, that $u > 0$ and $v = 0$ when $\pi + 2\phi > \theta > \pi - 2\phi$, and that $u = 0$ and $v < 0$ when $2\pi > \theta > \pi + 2\phi$. Since

$$u + iv = \frac{\partial \arg f(e^{i\theta})}{\partial\theta} - i\,\frac{\partial \log|f(e^{i\theta})|}{\partial\theta}$$

(any branch of $\arg f(e^{i\theta})$ $(2\pi > \theta > 0)$ can be chosen), it follows that f(E) is the complement of a set of points of the form $\{w\colon |w| \geq r,\ \pi - \alpha \leq \arg w \leq \pi + \alpha\}$, for some r and α. That the power series for f has real coefficients implies symmetry with respect to the real axis.

A straightforward but rather long computation yields the identity

$$\begin{aligned}\log\frac{f(z)}{z} &= \int_0^z [F(w) - 1]w^{-1}\,dw \\ &= 2\cos\phi\,\log\left[\left(\left(1 + \frac{4z\cos^2\phi}{(1-z)^2}\right)^{1/2} + \cos\phi\,\frac{1+z}{1-z}\right)(1+\cos\phi)^{-1}\right] \\ &\quad + 2\log 2\left[1 + z + ((1-z)^2 + 4z\cos^2\phi)^{1/2}\right]^{-1}.\end{aligned}$$

From the previous discussion of the boundary behavior of $F(z)$ it follows that $r = |f(-1)|$ and $\alpha = \arg[f(e^{i(\pi+2\phi)})/f(-1)]$, so that r is given by (1) and α has the value $\pi(1 - \cos\phi)$.

The function f is a limiting case of a class of functions considered by Goodman [1].

3. THE SOLUTION OF PROBLEM 2

The following four lemmas show that the function f in (2) solves Problem 2, for the value r given by (1).

LEMMA 1. *If* $1/4 < r < 1$, *Problem* 2 *has a solution* f *for which* $a_2 > 0$ *and* a_k *is real* $(k = 3, 4, \cdots)$.

Proof. Since S^* is compact and S_r^* is closed (that is, since the assumptions $S_r^* \supset \{g_n(z)\}_{n=1}^{\infty}$ and $g_n(z) \to g(z)$ imply that $g(z) \in S_r^*$), the problem has a solution.

Suppose $g(z) = z + a_2 z^2 + \sum_{n=3}^{\infty} b_n z^n$ solves Problem 2. Clearly, we may assume $a_2 > 0$. Also, it follows from the minimum principle that if $f \in S^*$, then $f \in S_r^*$ if and only if $|f(z)/z| \geq r$ $(z \in E)$. Define

$$\frac{z f'(z)}{f(z)} = \frac{1}{2}\frac{z g'(z)}{g(z)} + \frac{1}{2}\overline{\left(\frac{\bar{z} g'(\bar{z})}{g(\bar{z})}\right)},$$

where $f(z) = z + \sum_{n=2}^{\infty} c_n z^n$ $(z \in E)$. Then

$$f(z) = z\left(\frac{g(z)}{z}\right)^{1/2}\left(\frac{\overline{g(\bar{z})}}{z}\right)^{1/2} = z + a_2 z^2 + \sum_{n=3}^{\infty} a_n z^n,$$

where a_n is real for $n = 3, 4, \cdots$. The inequality $\Re\left(\frac{z f'(z)}{f(z)}\right) > 0$ $(z \in E)$ implies that $f \in S^*$. Since

$$|f(z)/z| = |g(z)/z|^{1/2}\,|\overline{g(\bar{z})}/z|^{1/2} > r \quad \text{for } z \in E,$$

we conclude that $f \in S_r^*$.

LEMMA 2. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *with* $a_2 > 0$ *solves Problem* 1, *then* $\liminf_{\rho \to 1} |f(\rho e^{i\phi})| = r$ *provided* $-\cos\phi > a_2/2$. *If* f *solves Problem* 2 *with* $a_2 > 0$ *and all coefficients real, then* $\lim_{\rho \to 1} f(-\rho) = -r$.

Proof. Suppose $\liminf_{\rho \to 1} f(\rho e^{i\phi}) \neq r$. If f solves Problem 1, there exists ρ_0 $(1 > \rho_0 > 0)$ such that $|f(\rho e^{i\phi})| > r$ for $1 > \rho > \rho_0$. Let

$$g(z) = \frac{z}{(1 + z e^{-i\phi})^2} \quad \text{and} \quad g_M(z) = M g^{-1}\left(\frac{g(z)}{M}\right) = z - 2e^{-i\phi}\left(1 - \frac{1}{M}\right) z^2 + \cdots,$$

where $M > 1$. The function g_M maps the disk E onto the disk $|z| < M$ with a rectilinear slit from $M e^{i\phi}$ to $\lambda e^{i\phi}$, where $4\lambda M^2 = (\lambda + M)^2$ [5, p. 224, Exercise 4]. We may choose M so that $\lambda/M > \rho_0$. Then the function

$$M f(g_M(z)/M) = z + \left(-2 e^{-i\phi}\left(1 - \frac{1}{M}\right) + \frac{a_2}{M}\right) z^2 + \cdots$$

belongs to S_r and

$$\left|-2 e^{i\phi}\left(1 - \frac{1}{M}\right) + \frac{a_2}{M}\right| \geq \left|\frac{a_2}{M} - 2\cos\phi\left(1 - \frac{1}{M}\right)\right|$$

$$= |a_2|\left|\frac{1}{M} - \frac{2\cos\phi}{|a_2|}\left(1 - \frac{1}{M}\right)\right| > |a_2|,$$

provided $-\cos\phi > |a_2|/2$. This contradicts the extremal property of f. If f solves Problem 2 and has real coefficients, then $f(-\rho)$ is real and $Mf(g_M(z)/M) \in S^*$ if $\phi = \pi$. A proof similar to the one above then shows that $f(-\rho) \to -r$ as $\rho \to 1$.

Now suppose $f(z) = z + a_2 z^2 + \cdots$ (all a_n real, and $a_2 > 0$) solves Problem 2. Since $f \in S_r^*$, we can represent f in the form

$$\frac{zf'(z)}{f(z)} = \int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\,d\mu(t), \qquad \text{where } \int_{-\pi}^{\pi} d\mu(t) = 1$$

and where μ is an increasing function of t $(-\pi \le t \le \pi)$. Since f has real coefficients, we may require that $\mu(t) = -\mu(-t)$. Hence

$$\frac{zf'(z)}{f(z)} = \int_{0}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\,d\mu(t) - \int_{-\pi}^{0} \frac{1+ze^{-it}}{1-ze^{-it}}\,d\mu(-t)$$

$$= \int_0^{\pi}\left[\frac{1+ze^{-it}}{1-ze^{-it}} + \frac{1+ze^{it}}{1-ze^{it}}\right]d\mu(t).$$

It then follows that

$$\log\frac{f(z)}{z} = -2\int_0^{\pi}\log(1-2z\cos t + z^2)\,d\mu(t), \tag{3}$$

the logarithmic function being chosen so that $\log 1 = 0$.

The following lemma is due to Keogh [3].

LEMMA 3. $\lim_{\rho\to 1} \arg f(\rho e^{i\theta}) = \pi[\mu(\theta^+) + \mu(\theta^-)]$ $(0 < \theta < \pi)$, *where the branch of the argument is chosen so that* $\arg(f(\rho)) = 0$ *for* $\rho > 0$.

We wish to show that for some $\phi < \pi$ the extremal function f satisfies the condition

$$\arg f(e^{i\theta}) = \lim_{\rho\to 1} \arg f(\rho e^{i\theta}) = \text{constant} \qquad (0 < \theta < \phi),$$

while

$$|f(e^{i\theta})| = \lim_{\rho\to 1} |f(\rho e^{i\theta})| = r \qquad (\phi < \theta \le \pi).$$

Suppose f does not have this property. By Lemmas 2 and 3, it is clear that there exist ψ_1 and ψ_2 with $\lim_{\rho\to 1} |f(\rho e^{i\theta})| \ne r$ whenever $0 < \psi_1 \le \theta \le \psi_2 < \pi$, and with $\mu(\psi_2^-) > \mu(\psi_1^+)$. Also, there exist values θ_1, θ_2, and θ_3 such that $(\psi_1 < \theta_1 < \theta_2 < \theta_3 < \psi_2)$ and

$$\mu(\theta_2^+) \ge \frac{1}{2}[\mu(\theta_3^-) + \mu(\theta_1^+)] \ge \mu(\theta_2^-),$$

where $\mu(\theta_3^-) > \mu(\theta_1^+)$. Without changing f as given by (3), we may redefine μ so that

$$\mu(\theta_1) = \mu(\theta_1^+), \qquad \mu(\theta_3) = \mu(\theta_3^-), \qquad \mu(\theta_2) = \frac{1}{2}[\mu(\theta_3^-) + \mu(\theta_1^+)].$$

With θ_1, θ_2, and θ_3 satisfying the conditions in the preceding paragraph, we now define a function

$$f_\varepsilon(z) = z + a_2(\varepsilon)z^2 + \cdots \in S_r^* \qquad \text{with } |a_2(\varepsilon)| > |a_2|.$$

For $0 < t < \pi$ and $1 > \varepsilon > -1$, choose $g(t, \varepsilon)$ so that $0 < g(t, \varepsilon) < \pi$ and

$$(4) \qquad (1+\varepsilon)\log\frac{2}{1+\cos g(t,\varepsilon)} = \log\frac{2}{1+\cos t}.$$

From this definition it follows that $\varepsilon > 0$ implies $g(t, \varepsilon) < t$ and $\varepsilon < 0$ implies $g(t, \varepsilon) > t$, and that $g(t, \varepsilon)$ is a strictly increasing function of t. Let $f_\varepsilon(z)$ be defined by the relation

$$\log[f_\varepsilon(z)/z] = -2\int_I \log(1 - 2z\cos t + z^2)\,d\mu(t)$$

$$(5) \qquad - 2(1+\varepsilon)\int_{\theta_1}^{\theta_2} \log(1 - 2z\cos g(t,\varepsilon) + z^2)\,d\mu(t)$$

$$- 2(1-\varepsilon)\int_{\theta_2}^{\theta_3} \log(1 - 2z\cos g(t,-\varepsilon) + z^2)\,d\mu(t),$$

where $I = [0, \theta_1] \cup [\theta_3, \pi]$ and where $0 \le \varepsilon < 1$.

LEMMA 4. *For sufficiently small* $\varepsilon > 0$, $f_\varepsilon(z) \in S_r^*$.

Proof. Since $\dfrac{\partial g(t,\varepsilon)}{\partial t} > 0$, there is an increasing function $v(t)$ with $v(-t) = -v(t)$ $(-\pi \le t \le \pi)$ such that $\int_{-\pi}^{\pi} dv(t) = 1$ and

$$\log[f_\varepsilon(z)/z] = -2\int_0^{\pi} \log(1 - 2z\cos t + z^2)\,dv(t).$$

This implies

$$\frac{z f_\varepsilon'(z)}{f_\varepsilon(z)} = \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\,dv(t),$$

so that $f_\varepsilon \in S^*$.

The relation

$$\frac{\partial \log |f_\varepsilon(z)/z|}{\partial \varepsilon}$$

$$= -2\left[\int_{\theta_1}^{\theta_2} \left(\log |1 - 2z \cos g(t, \varepsilon) + z^2| + (1+\varepsilon) \frac{\partial \log |1 - 2z \cos g(t, \varepsilon) + z^2|}{\partial \varepsilon} \right) d\mu(t) \right.$$

$$\left. + \int_{\theta_2}^{\theta_3} \left(-\log |1 - 2z \cos g(t, -\varepsilon) + z^2| + (1-\varepsilon) \frac{\partial \log |1 - 2z \cos g(t, -\varepsilon) + z^2|}{\partial \varepsilon} \right) d\mu(t) \right]$$

holds because the partial derivatives are uniformly bounded for each fixed z. Now let $z = \rho e^{i\theta}$, where either $0 \le \theta < \psi_1$ or $\psi_2 < \theta \le \pi$. If ε is sufficiently small, then $\psi_1 < g(\theta_1, \varepsilon)$ and $g(\theta_2, -\varepsilon) < \psi_2$, so that

$$\lim_{\rho \to 1} \frac{\partial \log |f_\varepsilon(z)/z|}{\partial \varepsilon}$$

$$= -2\left[\int_{\theta_1}^{\theta_2} \left(\log 2|\cos \theta - \cos g(t, \varepsilon)| + (1+\varepsilon) \frac{\partial \log 2|\cos \theta - \cos g(t, \varepsilon)|}{\partial \varepsilon} \right) d\mu \right.$$

$$\left. + \int_{\theta_2}^{\theta_3} \left(-\log 2|\cos \theta - \cos g(t, -\varepsilon)| + (1-\varepsilon) \frac{\partial \log 2|\cos \theta - \cos g(t, -\varepsilon)|}{\partial \varepsilon} \right) d\mu \right],$$

by virtue of uniform boundedness and continuity of the partial derivatives.

Therefore

$$\lim_{\rho \to 1} \frac{\partial \log |f_\varepsilon(z)/z|}{\partial \varepsilon} = -2\left[\int_{\theta_1}^{\theta_2} k(g(t, \varepsilon))\, d\mu - \int_{\theta_2}^{\theta_3} k(g(t, -\varepsilon))\, d\mu \right],$$

where $k(\psi) = \log 2|\cos \theta - \cos \psi| - \dfrac{1 + \cos \psi}{\cos \theta - \cos \psi} \log \dfrac{2}{1 + \cos \psi}$. Now we see that

$$k'(\psi) = \frac{\sin \psi\, (1 + \cos \theta)}{[\cos \theta - \cos \psi]^2} \log \frac{2}{1 + \cos \psi} > 0 \qquad (\pi \ne \theta \ne \psi,\ 0 \le \psi < \pi).$$

In the first integral above, $g(t, \varepsilon) < t \le \theta_2$, and in the second, $g(t, -\varepsilon) > t \ge \theta_2$, while θ is not in the interval $[g(\theta_1, \varepsilon), g(\theta_3, -\varepsilon)]$. Hence

$$\lim_{\rho \to 1} \frac{\partial \log |f_\varepsilon(z)/z|}{\partial \varepsilon} > -2\left[\int_{\theta_1}^{\theta_2} k(\theta_2)\, d\mu - \int_{\theta_2}^{\theta_3} k(\theta_2)\, d\mu \right] = 0.$$

This implies that for some $\rho_0 < 1$

$$\frac{\partial \log |f_\varepsilon(\rho e^{i\theta})/\rho e^{i\theta}|}{\partial \varepsilon} > 0 \qquad \text{whenever } 1 > \rho > \rho_0,$$

so that

$$|f_\varepsilon(\rho e^{i\theta})/\rho e^{i\theta}| \geq |f(\rho e^{i\theta})/\rho e^{i\theta}| \geq r \tag{6}$$
$$(1 > \rho > \rho_0 \text{ and } 0 < \theta < \psi_1 \text{ or } \psi_2 < \theta < \pi).$$

Now $\lim_{\rho \to 1} |f(\rho e^{i\theta})| \neq r$ for $\psi_1 \leq \theta \leq \psi_2$. Since $[\psi_1, \psi_2]$ is a closed interval, it follows that there exist $r_0 > r$ and $\rho_1 < 1$ such that

$$|f(\rho e^{i\theta})| \geq r_0 \quad \text{if } 1 > \rho \geq \rho_1 \text{ and } \psi_1 \leq \theta \leq \psi_2.$$

Since for $|z| \leq \rho < 1$, $f_\varepsilon(z)$ converges uniformly to $f(z)$ as $\varepsilon \to 0$, we see that $|f_\varepsilon(\rho e^{i\theta})| \geq r$, provided $\psi_1 \leq \theta \leq \psi_2$, $\rho_1 \leq \rho < 1$, and ε is sufficiently small. Using this together with (6), we conclude that $f_\varepsilon \in S_r^*$ if ε is sufficiently small ($\varepsilon > 0$).

THEOREM 1. *The solution of Problem* 2 *with* $a_2 > 0$ *is the function* $f \in S_r^*$ *given by* (2), *where* r *is given by* (1) *and* $a_2 = 2\cos^2\phi$. *The solution is unique, up to rotations.*

Proof. Suppose the solution is not as described in the theorem. Then, by Lemmas 3 and 4, the function $f_\varepsilon(z) = z + a_2(\varepsilon)z^2 + \cdots$ belongs to S_r^*, where f_ε is given by (5) and ε is sufficiently small ($\varepsilon > 0$).

Since

$$\log\frac{f_\varepsilon(z)}{f(z)} = [a_2(\varepsilon) - a_2]z + \cdots$$

$$= -2\int_{\theta_1}^{\theta_2} \log\frac{1 - 2z\cos g(t, \varepsilon) + z^2}{1 - 2z\cos t + z^2}\,d\mu - 2\int_{\theta_2}^{\theta_3} \log\frac{1 - 2z\cos g(t, -\varepsilon) + z^2}{1 - 2z\cos t + z^2}\,d\mu$$

$$- 2\varepsilon\left[\int_{\theta_1}^{\theta_2} \log(1 - 2z\cos g(t, \varepsilon) + z^2)\,d\mu - \int_{\theta_2}^{\theta_3} \log(1 - 2z\cos g(t, -\varepsilon) + z^2)\,d\mu\right],$$

it follows that

$$a_2(\varepsilon) - a_2 = 4\int_{\theta_1}^{\theta_2} [\cos g(t, \varepsilon) - \cos t + \varepsilon\cos g(t, \varepsilon)]\,d\mu$$

$$+ \int_{\theta_2}^{\theta_3} [\cos g(t, -\varepsilon) - \cos t - \varepsilon\cos g(t, -\varepsilon)]\,d\mu$$

$$= 4\varepsilon\left[\int_{\theta_1}^{\theta_2} h(t)\,d\mu - \int_{\theta_2}^{\theta_3} h(t)\,d\mu\right] + 2\varepsilon^2\left[\int_{\theta_1}^{\theta_2} k(t)\,d\mu + \int_{\theta_2}^{\theta_3} k(t)\,d\mu\right] + O(\varepsilon^3),$$

where

$$h(t) = (1 + \cos t)\log\frac{2}{1 + \cos t} + \cos t \quad \text{and} \quad k(t) = (1 + \cos t)\log^2\frac{2}{1 + \cos t}.$$

Since $h'(t) = -\sin t \log \frac{2}{1+\cos t} < 0$ for $0 < t < \pi$, we conclude that

$$a_2(\varepsilon) - a_2 \geq 4\left[\int_{\theta_1}^{\theta_2} h(\theta_2)\,d\mu - \int_{\theta_2}^{\theta_3} h(\theta_2)\,d\mu\right]$$

$$+ 2\varepsilon^2 \int_{\theta_1}^{\theta_3} (1+\cos t)\log^2 \frac{2}{1+\cos t}\,d\mu + O(\varepsilon^3) > 0$$

for sufficiently small values of ε ($\varepsilon > 0$). This proves that (2) gives a solution.

That the solution is unique follows from the fact that if g is extremal with $a_2 > 0$, then the function

$$f(z) = z\left(\frac{g(z)}{z}\right)^{1/2}\left(\frac{\overline{g(\bar{z})}}{\bar{z}}\right)^{1/2}$$

is also extremal and has real coefficients, and by the proof above, f must then be given by (2). But $\lim_{\rho\to 1} |f(\rho e^{i\theta})| = r$ if and only if the same is true for both $|g(z)|$ and $|\overline{g(\bar{z})}|$; moreover, $\arg f(e^{i\theta})$ is constant if and only if the same is true for $g(z)$ and $\overline{g(\bar{z})}$. This implies that $g(z) = f(z)$.

4. TWO CONJECTURES

Let $F(z)$ be normalized so that $F(0) = 0$ and $F'(0) = 1$, where

$$\frac{zF'(z)}{F(z)} = \frac{1}{\cos\phi}\frac{zf'(z)}{f(z)} + \left(1 - \frac{1}{\cos\phi}\right)\frac{f(z)}{zf'(z)} = \frac{1 + 4\cos\phi\dfrac{z}{(1-z)^2}}{\sqrt{1 + 4\cos^2\phi\dfrac{z}{(1-z)^2}}}, \tag{7}$$

and where f is given by (2). Letting $z = e^{i\theta}$ $(0 < \theta < \pi)$ and writing $\frac{e^{i\theta}F'(e^{i\theta})}{F(e^{i\theta})} = u + iv$, we see that

$$u = 0 \text{ and } v > 0 \quad \text{if } 0 < \sin^2\theta/2 < \cos^2\phi,$$

$$u < 0 \text{ and } v = 0 \quad \text{if } \cos^2\phi < \sin^2\theta/2 < \cos\phi,$$

$$u > 0 \text{ and } v = 0 \quad \text{if } \cos\phi < \sin^2\theta/2 .$$

From the discussion in Section 2 concerning the properties of f and $g \in S^*$, where $\frac{zg'(z)}{g(z)} = \frac{f(z)}{zf'(z)}$, we find that

$$\arg\frac{F(-1)}{F(e^{i(\pi-2\phi)})} = \int_{\pi-2\phi}^{\pi} u(\theta)\,d\theta$$

$$= \frac{1}{\cos\phi}\int_{\pi-2\phi}^{\pi} \frac{e^{i\theta}f'(e^{i\theta})}{f(e^{i\theta})}\,d\theta + \left(1 - \frac{1}{\cos\phi}\right)\int_{\pi-2\phi}^{\pi} \frac{f(e^{i\theta})}{e^{i\theta}f'(e^{i\theta})}\,d\theta$$

$$= \frac{1}{\cos\phi} \cdot \pi(1 - \cos\phi) + \left(1 - \frac{1}{\cos\phi}\right)\pi = 0 .$$

Since $F(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1}{\cos\phi}} \left(\frac{g(z)}{z}\right)^{1-\frac{1}{\cos\phi}}$, it follows that

$$F(-1) = -(r)^{\frac{1}{\cos\phi}} (M)^{1-\frac{1}{\cos\phi}} = -\frac{1}{(1+\cos\phi)^2},$$

where r is given by (1) and where $M = \dfrac{1}{1 - \cos^2\phi}$. Since F has real coefficients, we conclude that the image of E under the mapping F is the entire plane less the set of points

$$\left\{ w\colon w \le \frac{-1}{(1+\cos\phi)^2} \text{ or } |w| = \frac{1}{(1+\cos\phi)^2} \text{ and } \pi - \psi \le \arg w \le \pi + \psi \right\},$$

for some ψ. Further analysis shows that

$$\psi = \pi - 2\cos^{-1}\left[\frac{1 - \cos\phi}{1 + \cos\phi}\right].$$

Applying the principle of the argument to $1/F(z)$ and using the fact that $F(z) \neq 0$ if $z \neq 0$ and $z \in E$, we conclude that $F \in S$. An interesting extremal property of this mapping was proved by Goodman and Reich [2].

Conjecture 1. If $F(z) = z + a_2 z^2 + \cdots \in S_r$, then $|a_2| \le 8/\sqrt{r} - 6 - 2/r$, and equality is reached for the function F given by (7) with $r = \dfrac{1}{(1+\cos\phi)^2}$.

Lemma 2 seems to support this conjecture. Also, it is interesting to compare the conjecture with a result of Singh [6]. Singh considers the class of univalent functions $f(z) = \sum_1^\infty a_n z^n$ (all a_n real, $f(1) = 1$) that are regular in E and whose image domains cover E. He shows that in this class of functions the maximum of a_2 is attained by a function f that maps the disk onto the plane, minus the real axis from 1 to ∞ and from -1 to $-\infty$, and minus an arc of a circle symmetric about -1.

Lewin [4] introduced the class σ of biunivalent functions, which is defined as follows: $f \in \sigma$ if and only if $f \in S$ and there exists $g \in S$ such that $g(f(z)) = z$ in the disk $|z| < r$, for some $r > 0$. Lewin pointed out that if $f, g \in S_r$, then $\frac{1}{r} f(g^{-1}(rz)) \in \sigma$. If $f(z) = z + a_2 z^2 + \cdots$ and $g(z) = -f(-z)$, then

$$\frac{1}{r}(f(g^{-1}(rz)) = z + 2r a_2 z^2 + \cdots .$$

This, together with Conjecture 1 and the fact that $8\sqrt{r} - 6r - 2$ is largest when $r = 4/9$, suggests the following conjecture.

Conjecture 2. If $g(z) = z + a_2 z^2 + \cdots \in \sigma$, then $|a_2| \le 4/3$.

It is clear that $a_2 = 4/3$ for the function $g(z) = \frac{1}{r} F(-F^{-1}(-rz))$, where F is given by (7) and $r = \frac{1}{(1+\cos\phi)^2} = \frac{4}{9}$.

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