VON NEUMANN ALGEBRAS WITH A SINGLE GENERATOR

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A *von Neumann algebra* is a weakly closed, self-adjoint algebra of operators on a (complex) Hilbert space with the property that the identity operator on the Hilbert space belongs to the algebra. If $\{A_1, A_2, \cdots\}$ is a finite or countably infinite collection of operators acting on the Hilbert space $\mathfrak{H}$, then the *von Neumann algebra generated by the collection* $\{A_1, A_2, \cdots\}$ is by definition the smallest von Neumann algebra that contains each operator A_i; we denote this algebra by $\mathscr{R}(A_1, A_2, \cdots)$. It is known [1, p. 33] that if $\mathfrak{H}$ is separable, then every von Neumann algebra $\mathscr{A}$ acting on $\mathfrak{H}$ may be written as $\mathscr{A} = \mathscr{R}(A_1, A_2, \cdots)$ for some countable family $\{A_1, A_2, \cdots\}$ of operators in $\mathscr{A}$. A von Neumann algebra $\mathscr{A}$ is said to have a *single generator* if there exists an operator A in $\mathscr{A}$ such that $\mathscr{A} = \mathscr{R}(A)$. (It is easy to show that $\mathscr{A} = \mathscr{R}(A)$ if and only if $\mathscr{A}$ consists of the weak closure of the set of all polynomials $p(A, A^*)$ in A and A^*.)

Problem. Does every von Neumann algebra $\mathscr{A}$ acting on a separable Hilbert space have a single generator?

This problem has been before us for some time. The first result bearing on it is the theorem of von Neumann [4] that if $\mathscr{A}$ is abelian, then $\mathscr{A}$ has a single Hermitian generator. Further progress was made by Pearcy, who showed in [5] that $\mathscr{A}$ has a single generator if it is of type I, and who introduced in [6] a certain matricial technique that has turned out to be useful in subsequent work on this problem. Next, Suzuki and Saitô proved in [10] that if $\mathscr{A}$ is hyperfinite, then $\mathscr{A}$ has a single generator (see [3, footnote 68]). Finally, Wogen [11] recently extended certain important results of Saitô [9], and he used the extensions to prove that if $\mathscr{A}$ is properly infinite (that is, if $\mathscr{A}$ contains no nonzero finite central projection), then $\mathscr{A}$ has a single generator.

In most of these papers, von Neumann algebras $\mathscr{A}$ having the property

(T) $\mathscr{A}$ is (algebraically) * - isomorphic to the von Neumann algebra $M_2(\mathscr{A})$ of all 2×2 matrices over $\mathscr{A}$

play a central role.

The problem of identifying the von Neumann algebras with property (T) is difficult. In particular, it is known [3] that certain von Neumann algebras of type II_1 have property (T), but it is not known whether *every* von Neumann algebra of type II_1 has property (T).

The purpose of this note is to prove the following two theorems.

THEOREM 1. *Suppose that $\mathscr{A}$ is a von Neumann algebra of type* II_1 *acting on a separable, infinite-dimensional Hilbert space $\mathfrak{H}$. Suppose also that every von Neumann subalgebra of $\mathscr{A}$ that is of type* II_1 *(acting perhaps on a smaller space) has property* (T). *Then $\mathscr{A}$ has a single generator.*

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THEOREM 2. *If every von Neumann algebra of type* II_1 *acting on a separable Hilbert space has property* (T), *then the answer to Problem* 1 *is affirmative.*

LEMMA. *Let* n *be a positive integer, and suppose that for* $i = 1, 2, \cdots, n$, $\mathscr{A}_i$ *is a von Neumann algebra with a single generator* A_i. *Then the direct sum* $\mathscr{A} = \sum_{i=1}^{n} \oplus \mathscr{A}_i$ *has a single generator.*

Proof. It is easy to see that since A_i generates $\mathscr{A}_i$, the same is true of every operator of the form $A_i - \lambda$, where λ does not belong to the spectrum of A_i. Therefore we may assume that there exist mutually disjoint open discs $\Delta_1, \Delta_2, \cdots, \Delta_n$ in the complex plane such that the spectrum of A_i is contained in Δ_i $(1 \leq i \leq n)$. We define $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, and we show that A generates $\mathscr{A}$. Clearly, it suffices to fix an integer j $(1 \leq j \leq n)$ and to show that the operator

$$B = B_1 \oplus B_2 \oplus \cdots \oplus B_n$$

belongs to $\mathscr{R}(A)$, where $B_i = 0$ for $i \neq j$ and $B_j = A_j$. For this purpose, we employ a theorem of Lavrentiev [2], to assert the existence of a sequence of polynomials $\{p_n(z)\}$ that converges uniformly on every compact subset of $\bigcup_{i=1}^{n} \Delta_i$ to the analytic function

$$f(z) = \begin{cases} z & (z \in \Delta_j), \\ 0 & \left(z \in \left[\bigcup_{j=1}^{n} \Delta_i\right] - \Delta_j\right). \end{cases}$$

It follows (see [8, p. 432]) that the sequence $\{p_n(A)\}$ converges in the uniform operator topology to $f(A) = B$; since B clearly lies in $\mathscr{R}(A)$, the proof is complete.

Proof of Theorem 1. Let $\mathscr{A}$ satisfy the hypotheses of the theorem, and let ϕ be an algebraic * - isomorphism of $\mathscr{A}$ onto $M_2(\mathscr{A})$. We begin the argument by defining (for each positive integer n) a * - isomorphism ϕ_n of $\mathscr{A}$ onto the von Neumann algebra $M_{2^n}(\mathscr{A})$ of all $2^n \times 2^n$ matrices with entries from $\mathscr{A}$. (Obviously $M_{2^n}(\mathscr{A})$ acts on the direct sum of 2^n copies of $\mathfrak{H}$.) Let $\phi = \phi_1$. Assuming that ϕ_j has been defined for $1 \leq j \leq k$, let X be in $\mathscr{A}$, and suppose that

$$\phi_k(X) = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1,2^k} \\ X_{21} & X_{22} & \cdots & X_{2,2^k} \\ \cdots & \cdots & \cdots & \cdots \\ X_{2^k,1} & X_{2^k,2} & \cdots & X_{2^k,2^k} \end{pmatrix},$$

where each X_{ij} belongs to $\mathscr{A}$. We now define $\phi_{k+1}(X)$ to be the $2^{k+1} \times 2^{k+1}$ matrix

$$\phi_{k+1}(X) = \begin{pmatrix} \phi(X_{11}) & \phi(X_{12}) & \cdots & \phi(X_{1,2^k}) \\ \phi(X_{21}) & \phi(X_{22}) & \cdots & \phi(X_{2,2^k}) \\ \cdots & \cdots & \cdots & \cdots \\ \phi(X_{2^k,1}) & \phi(X_{2^k,2}) & \cdots & \phi(X_{2^k,2^k}) \end{pmatrix}.$$

By induction, we obtain a * - isomorphism ϕ_n: $\mathcal{A} \to M_{2^n}(\mathcal{A})$, for each positive integer n.

To show that $\mathcal{A}$ has a single generator, it suffices (by virtue of [11, Theorem 1]) to show that there exist operators A and B in $\mathcal{A}$ such that $\mathcal{A} = \mathcal{R}(A, B)$. The construction of these operators A and B is somewhat complicated, and it goes as follows. We know [1, p. 33] that there exists a countable collection $\{A_1, A_2, \cdots\}$ of operators in $\mathcal{A}$ such that $\mathcal{A} = \mathcal{R}(A_1, A_2, \cdots)$. For each positive integer k, let

$$\phi_k(A_k) = \begin{pmatrix} C_{11}(k) & C_{12}(k) & \cdots & C_{1,2^k}(k) \\ C_{21}(k) & C_{22}(k) & \cdots & C_{2,2^k}(k) \\ \cdots & \cdots & \cdots & \cdots \\ C_{2^k,1}(k) & C_{2^k,2}(k) & \cdots & C_{2^k,2^k}(k) \end{pmatrix},$$

where the entries $C_{ij}(k)$ belong to $\mathcal{A}$, and denote by $\mathcal{C}_k$ the von Neumann algebra

$$\mathcal{C}_k = \mathcal{R}(\{C_{ij}(k)\}_{i,j=1}^{2^k})$$

(acting on $\mathfrak{H}$). Clearly, $\mathcal{C}_k$ is contained in $\mathcal{A}$, and we know that $\mathcal{C}_k$ can be written as a direct sum $\mathcal{C}_k = \mathcal{E}_k \oplus \mathcal{F}_k$, where $\mathcal{E}_k$ and $\mathcal{F}_k$ are finite von Neumann algebras of types II and I, respectively. (Of course, either direct summand may be absent.) Since $\mathcal{C}_k$ is finitely generated, the same is true of the algebra $\mathcal{E}_k$, and since we assume that $\mathcal{E}_k$ has property (T), it follows from [11, Theorem 1] that $\mathcal{E}_k$ has a single generator. Since $\mathcal{F}_k$ has a single generator (by [5]), it follows from the lemma that $\mathcal{C}_k$ has a single generator—say $\mathcal{C}_k = \mathcal{R}(D_k)$. Multiplying by a suitable scalar, we may assume that $\|D_k\| \le k^{-2}$. The desired operator A will be constructed from the sequence $\{D_k\}$, and, roughly speaking, it will be the infinite matrix

$$\left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & \begin{array}{c|c} D_2 & 0 \\ \hline 0 & \begin{array}{c|c} D_3 & 0 \\ \hline 0 & \ddots \end{array} \end{array} \end{array}\right),$$

properly interpreted.

More precisely, we define the operator A as the sum of an infinite series $A = \sum_{n=1}^{\infty} T_n$ of operators T_n in $\mathcal{A}$. The sequence $\{T_n\}_{n=1}^{\infty}$ is defined as follows. Let T_1 be the unique operator in $\mathcal{A}$ such that

$$\phi(T_1) = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let T_2 be the unique operator in $\mathscr{A}$ such that

$$\phi_2(T_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In general, for each positive integer n, we define T_n as the unique operator in $\mathscr{A}$ for which the matrix $\phi_n(T_n)$ in $M_{2^n}(\mathscr{A})$ has D_n as its $(2^n - 1, 2^n - 1)$th entry and zeros elsewhere. Since each isomorphism ϕ_n is norm-preserving [1, p. 8], we have the inequality

$$\|T_n\| = \|D_n\| \leq n^{-2} \quad (1 \leq n < \infty),$$

and thus the series $\sum_{n=1}^{\infty} T_n$ converges in the uniform operator topology to some operator A in $\mathscr{A}$.

We assert that for each positive integer n, the $(2^n - 1, 2^n - 1)$th entry of the matrix $\phi_n(A)$ is the operator D_n. To prove this, it suffices to show that for every positive integer k different from n the $(2^{n-1}, 2^{n-1})$th entry of the matrix $\phi_n(T_k)$ is 0. The argument splits into the two cases $k > n$ and $k < n$. Suppose first that $k < n$. Then the $(2^k, 2^k)$th entry of the matrix $\phi_k(T_k)$ is 0, and this implies that both the $(2^{k+1} - 1, 2^{k+1} - 1)$th and the $(2^{k+1}, 2^{k+1})$th entries of the matrix $\phi_{k+1}(T_k)$ are 0. By a finite induction argument, one readily concludes that the $(2^n - 1, 2^n - 1)$th entry of the matrix $\phi_n(T_k)$ is 0. Suppose now that $k > n$, and suppose, contrary to our assertion, that the $(2^n - 1, 2^n - 1)$th entry of $\phi_n(T_k)$ is not 0. Then at least one of the four entries with indices

$$(2^{n+1} - 2, 2^{n+1} - 2), \quad (2^{n+1} - 2, 2^{n+1} - 3), \quad (2^{n+1} - 3, 2^{n+1} - 2), \quad (2^{n+1} - 3, 2^{n+1} - 3)$$

in the matrix $\phi_{n+1}(T_k)$ is not 0. Again by a finite induction argument, we conclude that the matrix $\phi_k(T_k)$ contains at least one nonzero entry in a position other than the $(2^k - 1, 2^k - 1)$th position; this is a contradiction. Thus we have shown that for each positive integer n, the $(2^n - 1, 2^n - 1)$th entry of the matrix $\phi_n(A)$ is D_n.

We next show that there exists an operator B in $\mathscr{A}$ such that $\mathscr{A} = \mathscr{R}(A, B)$. To obtain B, we first construct a hyperfinite subfactor $\mathscr{B}$ of $\mathscr{A}$ as follows. Let $\mathscr{B}_1 \subset \mathscr{A}$ be the subfactor of type I_2 such that $\phi(\mathscr{B}_1)$ consists of all linear combinations of the four matrix units

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In general, for each positive integer n, let $\mathscr{B}_n$ be the type I_{2^n} subfactor of $\mathscr{A}$ such that $\phi_n(\mathscr{B}_n)$ consists of all linear combinations of the 2^{2n} different matrices in

$M_{2^n}(\mathscr{A})$, each having one entry equal to $1_{\mathfrak{H}}$ and all other entries equal to 0. Note that

$$\mathscr{B}_1 \subset \mathscr{B}_2 \subset \mathscr{B}_3 \subset \cdots \subset \mathscr{B}_n \subset \cdots .$$

If $\mathscr{B}$ denotes the smallest von Neumann algebra containing each of the factors $\mathscr{B}_n$ $(1 \leq n < \infty)$, then $\mathscr{B}$ is a hyperfinite subfactor of $\mathscr{A}$ [1, p. 290]. Therefore, by virtue of [10, Theorem 1], it follows that $\mathscr{B} = \mathscr{R}(B)$ for some operator B in $\mathscr{B}$.

We complete the proof of the theorem by showing that $\mathscr{R}(A, B) = \mathscr{A}$. Since $\mathscr{A} = \mathscr{R}(A_1, A_2, \cdots)$, it suffices to show that each operator A_n lies in $\mathscr{R}(A, B)$. Since each ϕ_n is an isomorphism, this can be accomplished by showing that $\phi_n(A_n)$ lies in $\phi_n(\mathscr{R}(A, B))$ $(1 \leq n < \infty)$. To this end, fix a positive integer n, and note first that

$$\phi_n(\mathscr{R}(A, B)) \supset \phi_n(\mathscr{R}(B)) = \phi_n(\mathscr{B}) \supset \phi_n(\mathscr{B}_n).$$

Thus every $2^n \times 2^n$ matrix having a $1_{\mathfrak{H}}$ for one entry and all other entries equal to 0 lies in $\phi_n(\mathscr{R}(A, B))$. Since $\phi_n(A)$ is in $\phi_n(\mathscr{R}(A, B))$, since the $(2^n - 1, 2^n - 1)$th entry of $\phi_n(A)$ is D_n, and since $\mathscr{C}_n = \mathscr{R}(D_n)$, we see that $\phi_n(\mathscr{R}(A, B))$ contains the algebra $M_{2^n}(\mathscr{C}_n)$ and therefore the matrix $\phi_n(A_n)$. Thus the proof is complete.

Proof of Theorem 2. If $\mathscr{A}$ is a von Neumann algebra acting on a separable Hilbert space, then we can write $\mathscr{A}$ as the direct sum $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \mathscr{A}_3$, where $\mathscr{A}_1$ is a finite von Neumann algebra of type I, $\mathscr{A}_2$ is a von Neumann algebra of type II_1, and $\mathscr{A}_3$ is a properly infinite von Neumann algebra. By [5], $\mathscr{A}_1$ has a single generator, by Theorem 1 $\mathscr{A}_2$ has a single generator, and by [11, Theorem 2], $\mathscr{A}_3$ has a single generator. Thus we conclude from the lemma that $\mathscr{A}$ has a single generator.

Remark. We wish to acknowledge the contribution made by Professor David Topping toward the solution of Problem 1. Topping directed Wogen's thesis [11], and he was co-author of the joint paper with Pearcy [7] that led to the important paper of Saitô [9].

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