ON THE MILNOR-SPANIER AND ATIYAH DUALITY THEOREMS

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The theorems of Milnor and Spanier [5] and of Atiyah [1] state that the Thom spaces of certain pairs of vector bundles over a manifold are (stably) dual to each other in the sense of Spanier and Whitehead. In this note, we give a slight but natural generalization of these theorems, and we relate the result to Poincaré duality. For a different approach, see P. Holm [2].

THEOREM. *Let* M *be an* m*-dimensional compact manifold whose boundary* ∂M *is the union of two manifolds* A *and* B *with* $\partial A = \partial B = A \cap B$, *and suppose that* M *is embedded in* R^m (*the same* m). *Then* M/A *is* m*-dual to* M/B.

Here M/A means M with A collapsed to a point; R^m is real m-space. Two (reasonable) spaces are m-*dual* if they can be embedded (up to homotopy type) disjointly in the sphere S^{m+1} in such a way that for each the inclusion in the complement of the other is a stable homotopy equivalence (it is, in fact, sufficient that this hold for one of them).

The proof goes along the lines of [1] and [5]. There are the inclusions $M \subset R^m = R^m \times 0 \subset R^{m+1}$. Define $T = \{x \in R^{m+1}: x_{m+1} \geq 1\}$ and, as usual, $I = [0, 1] \subset R$. The subset $P = M \times 0 \cup A \times I \cup T$ of R^{m+1} is easily seen to be of the homotopy type of M with a cone erected over A (T is contractible, and the contraction extends to a deformation of P; thus we may collapse it to a point) and thus of the homotopy type of M/A, since M and A are ANR's. The same is true for the (closed) subset P' of S^{m+1} obtained from P by addition of the point at infinity.

We show next that the complement $Q = S^{m+1} - P' = R^{m+1} - P$ is of the homotopy type of M/B. We write Q as union of two subsets C and D, defined by

$$C = (M - A) \times (0, 1) \quad \text{and} \quad D = (R^{m+1} - (T \cup M \times I)) \cup (B^\circ \times (0, 1));$$

here $(0, 1)$ is the open interval, and B° means the interior $B - \partial B$. One sees easily that (i) D is contractible and the contraction extends to a deformation of Q, so that Q is of the homotopy type of Q/D; (ii) C is of the homotopy type of M; and (iii) Q/D $[= C/(B^\circ \times (0, 1))]$ is of the homotopy type of M/B.

For this, we use an exterior collar of ∂M, that is, a homeomorphism f of $\partial M \times I$ onto a neighborhood of ∂M in $R^m - M^\circ$, with $f(y, 0) = y$; also a collar of B in M that over ∂B gives a collar of ∂B in A; and a similar collar for A. To prove the existence of the exterior collar, we may have to shrink M, using an interior collar (which exists, by M. Brown's theorem). Now for the constructions: The first term of D has an obvious contraction, which begins by pushing down until $x_{m+1} < 0$. To contract all of D, we first pull $B^\circ \times (0, 1)$ out of $M \times I$, by moving all points $(f(y, t), x_{m+1})$ of D in the t-direction, so that the (t, x_{m+1})-unit square is contracted onto the triangle $x_{m+1} \leq t$.

The open set Q contains a copy, say Q', of M with a cone erected over B: we shrink $M \times 1/2$ slightly, using the collar over A, and contract B to a point in D. The inclusion $Q' \subset Q$ is a stable homotopy equivalence (one sees easily, *via*

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Mayer-Vietoris, that the homology is preserved; Q' is even a deformation retract of Q). The two sets P' and Q' clearly realize the m-duality asserted in Theorem 1.

We note that P' contains a copy of M with a cone over A attached, as deformation retract; the cone over A comes from the contraction of $A \times 1$ in T.

As a corollary we get the theorem of Atiyah:

COROLLARY. *Let the closed compact manifold* M *be embedded in* R^t; *suppose the normal bundle* ν *of* M *in* R^t *splits into the direct sum of two bundles* ξ, η. *Then the Thom spaces of* ξ *and* η *are* t-*dual.*

Proof. The Whitney sum of the ball bundles E and F of ξ and η is a t-manifold N in R^t. The boundary of this manifold falls naturally into two parts A and B; here A is the sphere bundle of the lift of E to F, and similarly for B. We are in the situation of the theorem, and so N/A and N/B are t-dual. But clearly N/A contains the Thom space of ξ as deformation retract, and similarly for η. Atiyah's first proposition [1, (3.2)] for a manifold with boundary has a similar proof.

We now connect with Poincaré duality. Let M again be as in the theorem ($\dim M = m$, $\partial M = A \cup B$ with $\partial A = \partial B = A \cap B$), and embed M in some Euclidean space R^{m+k} (we now allow $k > 0$). Let ν be the normal bundle of M, and let N be its ball bundle. We divide the boundary of N into two parts G and H: we take G equal to $N \mid A$, the ball bundle of $\nu \mid A$, and H equal to the union of the sphere bundle of ν and of $N \mid B$, the ball bundle of $\nu \mid B$. The theorem guarantees $(m+k)$-duality between N/G and N/H. Clearly, N/G has M/A as deformation retract. On the other hand, N/H is the relative Thom space of the bundle ν over the pair (M, B). Therefore there exists the Thom isomorphism between the cohomology $H^i(M, B)$ (with twisted coefficients, if M is not orientable) and the (reduced) cohomology $\tilde{H}^{i+k}(N/H)$. By Alexander duality (in S^{m+k+1}), the latter is isomorphic to the homology $\tilde{H}_{m-i}(N/G)$ or $H_{m-i}(M, A)$. Thus we have arrived at Poincaré duality in its general relative form.

The arguments above with bundles are standard, if M is a differentiable manifold. If M is merely topological, we use the results of Milnor on the existence of (stable) normal microbundles [4], together with the Kister-Mazur theorem that a microbundle always contains a vector bundle [3].

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