ASSOCIATED FIBRE SPACES

James D. Stasheff

From the point of view of homotopy theory, this paper looks at transformation groups and the association between G-bundles and principal G-bundles. It makes fundamental extensions to a theory of "transformation monoids," and it discusses a correspondence between quasifibrations and associated principal quasifibrations.

1. INTRODUCTION

In the theory of transformation groups, orbit spaces play an important role. It is particularly helpful if the map $X \to X/G$ is a fibre bundle, in fact, a principal G-bundle. In the theory of fibre bundles, the correspondence between G-bundles with fibre F and associated principal G-bundles is of crucial importance. If $p: E \to B$ is a (right) principal G-bundle and G is represented as a transformation group on F, the associated G-bundle $q: E \times_G F \to B$ with fibre F is defined by $E \times_G F = E \times F/G$, where G acts via the diagonal action $g(e, f) = (eg^{-1}, gf)$. If G is a transformation group on X and $X \to X/G$ is not a bundle, we can study the Borel bundle $\mathcal{E}_G \times X \to \mathcal{E}_G \times_G X = X_G$ [2], where $\mathcal{E}_G \to B_G$ is a universal principal G-bundle. [We use the notation $\mathcal{E}_G$, in contrast to [2] and [4], so that E_G can refer unambiguously to the above construction with X = E. There is no space $\mathcal{E}$ in this paper.] The total space $\tilde{X} = \mathcal{E}_G \times X$ has the same weak homotopy type as X, and if $X \to X/G$ were a principal G-bundle, X_G would have the same weak homotopy type as X/G.

Once we have adopted the point of view of weak homotopy type, it is natural to consider fibre spaces and even quasifibrations [5] instead of bundles, and weak homotopy equivalences instead of homeomorphisms. Since weak homotopy equivalences do not have precise inverses, it is appropriate to look at monoids of weak homotopy equivalences rather than groups thereof. This paper is an initial contribution to the theory of *transformation monoids* with particular emphasis on the role of (quasi-) fibrations.

We begin by fixing some elementary notation and terminology.

Definition 1.1. Let M be a topological monoid. A (left) M-*space* $(X; \mu)$ consists of a space X and a map $\mu: M \times X \to X$ such that (with $\mu(m, x) = mx$)

$$1)\ m(nx) = (mn)x, \qquad 2)\ ex = x.$$

If $\mu(m, \ ): X \to X$ is a weak homotopy equivalence of X for each $m \in M$, we say $(X; \mu)$ is a *representation of* M *by weak homotopy equivalences of* X or simply a *weak representation of* M. The adjoint to μ is a function $M \to X^X$, and it is a homomorphism. For a *right* M-space, the adjoint is an antihomomorphism. We then speak of a *weak antirepresentation of* M.

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For an M-space $(X; \mu)$, the relation $x \sim mx$ is not necessarily an equivalence relation on X. This poses problems in defining an orbit space. (If Mx denotes $\{mx \mid m \in M\}$, then $y \in Mx$ need not imply $x \in My$.) The relation does of course generate an equivalence relation that can be used to define an orbit space [6], but our interest will be in constructing an associated principal quasifibration $\tilde{X} \to X_M$. The construction (see Section 3) will be functorial and will agree up to weak homotopy with the construction for transformation groups. The construction is a generalization of that due to Dold and Lashof [4] for the special case in which X is a point, giving the universal principal quasifibration $\mathcal{E}_M \to B_M$.

The classical construction of X_G as $\mathcal{E}_G \times_G X$ is a special case of the general construction of $E \times_G X \to B$. This construction is not possible for monoids, as it stands, but our construction applies appropriately to a principal quasifibration $r: X \to A$ over M and a weak representation $(F; \nu)$ of M. We obtain a quasifibration $q: X \tilde{\times}_M F \to X_M$ and a weak homotopy equivalence $\rho: X_M \to A$. This associated fibre space behaves well in that it is functorial in the appropriate way and satisfies Theorems A, B, and C below.

For each quasifibration $q: D \to A$, we define an associated map Prin q: Prin D $\to$ A by

$$\text{Prin } D = \{\phi: F \to D \mid \phi \text{ is a weak homotopy equivalence with some } q^{-1}(b)\}$$

and Prin $q(\phi) = q(\phi(F))$. The typical "fibre" looks like the monoid $\mathcal{H}(F)$ of all weak homotopy equivalences of F into itself. In order to ensure that $\mathcal{H}(F)$ acts continuously on Prin D, we assume that F is locally compact, and we use the compact-open topology. It is not known whether Prin q is a quasifibration, except in the case of bundles and Hurewicz fibrations.

Definition 1.2. Two quasifibrations $r: C \to A$ and $r': C' \to A'$ are *quasi-equivalent* if there exists a fibre-preserving map

$$\begin{array}{ccc} C & \xrightarrow{\bar{f}} & C' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

such that f and $\bar{f}$ are weak homotopy equivalences. If r and r' are principal quasifibrations over M (see Section 3 for the definition), they are *structurally equivalent* if $\bar{f}$ can be chosen so that in addition $\bar{f}(cm) = \bar{f}(c)m$.

THEOREM A. *If* $q: D \to A$ *is a quasifibration with fibre* F *and* Prin q *is a quasifibration, then* q *is quasi-equivalent to* Prin $D \tilde{\times}_{\mathcal{H}(F)} F \to (\text{Prin } D)_{\mathcal{H}(F)}$.

THEOREM B. *If* $r: C \to A$ *is a principal quasifibration over* $\mathcal{H}(F)$, *then* Prin$(C \tilde{\times}_{\mathcal{H}(F)} F)$ *is structurally equivalent to* C.

Since every quasifibration is quasi-equivalent to a Hurewicz fibration, these theorems show that in the weak homotopy category there is a complete equivalence between quasifibrations with fibre F (locally compact) and associated principal quasifibrations over $\mathcal{H}(F)$, much as in the Steenrod theory of fibre bundles.

Special cases of the construction given here were considered in [15]. There the emphasis was on a generalization involving nontransitive representation of a monoid H by homotopy equivalences, in other words, involving a map $H \to H(F)$ that is not a strict homomorphism, but only an appropriate homotopy analogue. Here we emphasize the analogies with classical bundle theory, as in the above theorems and in the following one.

THEOREM C. *If* G *is a transformation group on* F *and* p: $E \to B$ *is a principal* G-*bundle, then* $E \tilde{\times}_G F \to E_G$ *is a* G-*bundle, and as such it is equivalent to* $\rho^* E \times_G F$, *where* $\rho: E_G \to B$ *is the map induced by* p.

Thus we have a satisfactory generalization. Our techniques use locally compact fibres, and they give rise to weak equivalences and quasifibrations. Some of the results could be strengthened with more elaborate machinery;* but the construction given here is already fairly complex. Since it suffices for all applications involving singular homotopy theory, we avoid further elaboration. At certain points, we assume familiarity with the Dold-Lashof construction, at least with its definition and conceptual significance. Otherwise the paper is self-contained; but it is closely related to others involving essentially the same kind of construction [15], [13], [7], [11].

A particularly important application involving singular homotopy theory is the Eilenberg-Moore spectral sequence [10], which converges to a graded group associated with $H^*(E \times_G F; k)$. Under suitable restrictions on the cohomology of the space (for example, if k is a field), the E_2-term is identifiable as

$$\mathrm{Coext}_{H^*(G;k)}\,(H^*(E; k),\, H^*(F; k)).$$

Eilenberg and Moore derive this spectral sequence by using differential homological algebra on the chain complexes involved. Several people have noted that the constructions by Dold and Lashof and by Milnor give geometric realizations of this spectral sequence in the special cases where E is universal and G is a group [1] or F is a point [7], [11]. Our construction substantiates the hint in [1] that the application is possible for arbitrary monoids M, principal quasifibrations over M, and weak representations of M. We sketch details in an appendix, where we also consider alternate forms of the construction and clear up problems about homotopy type depending on the topologies used.

We are grateful to John Derwent for pointing out these latter difficulties, casually passed over elsewhere in the literature. An earlier version of this paper was concerned primarily with the above realization. For the present emphasis, we are indebted to the insight of the referee.

2. THE BASIC CONSTRUCTION

Given a q.f. p: $E \to B$ in which H operates, Dold and Lashof imbed it in a q.f. $\hat{p}: \hat{E} \to \hat{B}$ such that the inclusion $E \subset \hat{E}$ is null-homotopic. In the principal case, this leads by iteration to a total space with the weak homotopy type of a point. To obtain a total space of the weak homotopy type of X, we alter the construction of Dold and Lashof.

Definition 2.1. A map q: $D \to A$ *is a quasifibration* if

$$q_*: \pi_i(D, q^{-1}(a)) \to \pi_i(A, a)$$

is an isomorphism for all $i \geq 0$. If A is path-connected, all the fibres $q^{-1}(a)$ have the same weak-homotopy type, often denoted by F.

Definition 2.2 (see [4]). An *operation* of M *in a quasifibration* r: $C \to A$ is a map $\nu: C \times M \to C$ such that, with the notation $\nu(c, m) = cm$,

* Added in proof: Compare N. E. Steenrod, *Milgram's classifying space for a topological group*, Topology (to appear).

a) $ce = c$ (ν has e as unit),

b) $r(cm) = r(c)$ (ν preserves fibres),

c) $\nu(c, \ ): M \to r^{-1}(r(c))$ is a weak homotopy equivalence.

Our basic construction is applicable if we are given

1) a fixed right M-space (X, μ),

2) an operation ν of M in a quasifibration $r: C \to A$, and

3) a map $\phi: C \to X$ such that $\phi(cm) = \phi(c)m$.

We shall imbed r in a map $\hat{r}: \hat{C} \to \hat{A}$. Except for the topologies involves,

$$\hat{C} = C \underset{\nu}{\cup} C \times I \times M \underset{\phi\times 1}{\cup} X \times M \quad \text{and} \quad \hat{A} = A \underset{r}{\cup} C \times I \underset{\phi}{\cup} X,$$

where the symbols denote identifications with respect to the maps indicated at $t = 0$ or 1, respectively, and where the map $\hat{r}: \hat{C} \to \hat{A}$ is induced by r or by projection onto all but the last factor on the respective pieces.

We use the same notational conventions as Dold and Lashof. A point of $\hat{C}$ is denoted by $c\,|t|\,m$, and a point of $\hat{A}$ by $c \perp t$. Note that $c\,|1|\,m = c'\,|1|\,m'$ if $cm = c'm'$, and $c\,|0|\,m = c'\,|0|\,m$ if $\phi(c) = \phi(c')$. The topology in $\hat{C}$ is the strongest topology such that the coordinate functions

$$\begin{aligned} t&: \hat{C} \to [0, 1] & c\,|t|\,m &\to t,\\ c&: t^{-1}(0, 1) \to C & c\,|t|\,m &\to c,\\ m&: t^{-1}(0, 1) \to M & c\,|t|\,m &\to m,\\ cm&: t^{-1}(0, 1] \to C & c\,|t|\,m &\to cm,\\ \phi(x)&: g^{-1}[0, 1) \to X & c\,|t|\,m &\to \phi(c) \end{aligned}$$

are continuous.

Only the last condition is not given explicitly by Dold and Lashof. Their construction is a special case of ours, namely the case where X is a point.

For $\hat{A}$, the coordinate functions that determine the topology are

$$\begin{aligned} t&: \hat{A} \to [0, 1] & c \perp t &\to t,\\ c&: t^{-1}(0, 1) \to C & c \perp t &\to c,\\ r(c)&: t^{-1}(0, 1] \to A & c \perp t &\to r(c),\\ \phi(c)&: t^{-1}[0, 1) \to X & c \perp t &\to \phi(c). \end{aligned}$$

The main properties of this construction are described in the following theorem.

THEOREM 2.3. *The map* $\hat{r}$ *is a quasifibration. There is a commutative diagram*

$$\begin{array}{ccc} C & \subset & \hat{C} \\ r\downarrow & & \downarrow\hat{r}, \\ A & \subset & \hat{A} \end{array}$$

where $C = \hat{r}^{-1}(A)$, *and* ϕ *extends to a map* $\hat{\phi}: \hat{C} \to X$.

Proof. That $\hat{r}$ is a quasifibration follows by reasoning similar to that in [4]. The map $\hat{\phi}: \hat{C} \to X$ is defined by $c\,|t|\,m \to \phi(cm)$. The equivariance of ϕ shows $\hat{\phi}$ is well-defined and continuous, since the coordinate functions m, cm, and $\phi(c)$ are continuous on the open sets on which they are defined.

3. PRINCIPAL QUASIFIBRATIONS

Definition 3.1 (see [4]). A principal *quasifibration over* M consists of an operation μ of M in a quasifibration $r: C \to A$ such that $(cm)m' = c(mm')$. (It follows that (C, μ) is a weak antirepresentation of M.)

THEOREM 3.2. *If* $r: C \to A$ *is a principal quasifibration with a map* $\phi: C \to X$ *such that* $\phi(cm) = \phi(c)m$ *for a fixed right* M-*space* X, *then* $\hat{r}: \hat{C} \to \hat{A}$ *is a principal quasifibration with* $\hat{\mu}: \hat{C} \times M \to \hat{C}$ *defined by* $\hat{\mu}(c\,|t|\,m, m') = c\,|t|\,mm'$, *and* $\hat{\phi}\hat{\mu}(\hat{c}, m) = \hat{\mu}(\hat{\phi}(\hat{c}), m)$.

The proof is the same as in [4], except for the last statement, which follows from $\hat{\phi}(c\,|t|\,m) = \phi(c)m$.

Thus, if $r: C \to A$ is a principal quasifibration over M with a map $\phi: C \to X$ such that $\phi(cm) = \phi(c)m$, then our construction can be iterated. We topologize the limit $r_\infty: C_\infty \to A_\infty$ just as Dold and Lashof do. A_∞ is to have the limit topology, but we give C_∞ a stronger topology, which can be described verbatim as in Dold and Lashof [4].

THEOREM 3.3. $r_\infty: C_\infty \to A_\infty$ *is a principal quasifibration over* M.

For Dold and Lashof, the limit total space is acyclic. Comparable results are obtained in our situation in two cases. First, we suppose that C is X itself, in other words, that $r: X \to A$ is a principal quasifibration over M and ϕ is the identity map on X.

THEOREM 3.4. *The equivariant imbedding*

$$\begin{array}{ccc} X & \longrightarrow & X_\infty \\ r\downarrow & & \downarrow r_\infty \\ A & \longrightarrow & A_\infty \end{array}$$

exhibits $X \to A$ *as an equivariant deformation retract of* $X_\infty \to A_\infty$. *The retraction* $X_\infty \to X$ *is given by* ϕ_∞.

Proof. By induction, let $C_{n+1} = C_n \cup D_n \times I \times M \cup X \times M$, and let $\phi_n: C_n \to X$ be an equivariant deformation retraction. It follows that $\phi_{n+1}(c\,|t|\,m) = \phi_n(c)m$ is an equivariant retraction. That it is an equivariant deformation retraction follows by the usual argument [12, Section 1.4, Lemma 9] from the fact that C_{n+1} can be equivariantly deformed into X, namely by $c\,|t|\,m \to \phi(c)\,|t + (1 - t)(1 - s)|\,m$.

For quasifibrations, there are various notions of equivalence. Those given in the introduction are particularly appropriate for our work here.

Definition 1.2. Two quasifibrations $r: C \to A$ and $r': C' \to A'$ are *quasi-equivalent* if there exists a fibre-preserving map

$$\begin{array}{ccc} C & \xrightarrow{\bar{f}} & C' \\ r\downarrow & & \downarrow r' \\ A & \xrightarrow{f} & A' \end{array}$$

such that f and $\bar{f}$ are weak homotopy equivalences. If in addition r and r' are principal quasifibrations, we say they are *structurally equivalent* if $\bar{f}$ can be chosen so that $\bar{f}(cm) = \bar{f}(c)m$.

Remark. In place of the homotopy condition on f or $\bar{f}$, we can assume that $\bar{f}$ induces a weak equivalence between corresponding fibres $r^{-1}(a)$ and $r'^{-1}(f(a))$.

Theorem 3.4 has the following particular consequence.

COROLLARY 3.5. r *and* r_∞ *are structurally equivalent.*

Now if X is not given as the total space of a principal quasifibration (for example, if no orbit space is defined, or if $X \to X/M$ is not a quasifibration), we consider $r: X \times M \to X$ by projection on the first factor, with $\phi = \mu: X \times M \to X$.

This special case is so important that we denote C_∞ by $\tilde{X}$ and A_∞ by X_M.

THEOREM 3.6. *If* (X, μ) *is a weak antirepresentation of* M, *then* $\tilde{X}$ *has the weak homotopy type of* X.

(What we have constructed resembles a free resolution of X over M.)

Proof. The map $X \to *$ (where $*$ denotes a point) is equivariant and hence induces maps

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \mathcal{E}_M \\ \downarrow & & \downarrow \\ X_M & \longrightarrow & B_M \end{array},$$

where $\mathcal{E}_M \to B_M$ is the universal principal quasifibration over M given by the Dold and Lashof construction on $M \to *$. Since the fibre of $\tilde{X} \to \mathcal{E}_M$ is X, the theorem follows from the next proposition.

PROPOSITION 3.7. *If* (X, μ) *is a weak antirepresentation of* M, *then* $C_n \to \mathcal{E}_n$ *is a quasifibration.* (Here $\mathcal{E}_n$ is the nth stage of the Dold and Lashof construction.)

Proof. The usual arguments apply. Notice that $C_{n+1} = C_n \cup C_n \times I \times M \cup X \times M$ and $\mathcal{E}_{n+1} = \mathcal{E}_n \cup \mathcal{E}_n \times I \times M \cup M$. We need the fact that $\phi: C_n \to X$ induces weak homotopy equivalences on the fibres of $C_n \to \mathcal{E}_n$. Since $\phi(c\,|t|\,m) = \phi(cm) = \phi(c)m$, this follows inductively from the case $\phi = \mu: X \times M \to X$, because μ is an antirepresentation by weak homotopy equivalences.

If $r: X \to A$ is a principal quasifibration over M, we might try to compare $\tilde{r}: \tilde{X} \to X_M$ and $r_\infty: X_\infty \to A_\infty$. We have the map

$$\begin{array}{ccc} X \times M & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle r} \\ X & \xrightarrow[r]{} & A \end{array}$$

Naturality of the construction gives

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X_\infty \\ \downarrow & & \downarrow \\ X_M & \longrightarrow & A_\infty \end{array}$$

THEOREM 3.8. *$\tilde{r}$: $\tilde{X} \to X_M$ and r_∞: $X_\infty \to A_\infty$ are structurally equivalent.*

Proof. The inclusion $X \to X_\infty$ factors through $\tilde{X}$ by identification of X with $X \times e$. The map $X \to \tilde{X}$ has already been shown to be a weak homotopy equivalence; therefore $\tilde{X} \to X_\infty$ is also a weak homotopy equivalence. Since the corresponding fibres of π_1 and r are mapped by weak homotopy equivalences, the same is true for the fibres of $\tilde{X} \to X_M$ and $X_\infty \to A_\infty$.

4. THE BASIC ASSOCIATED CONSTRUCTION

We turn to the problem of associated fibrations. Suppose we are given a left M-space F and a principal quasifibration $r: X \to A$ over M. If M is a group G, we can define the associated G-bundle $X \times_G F \to A$ with fibre F by means of a diagonal action of G on $X \times F$; but this uses inverses. As an alternative, we modify our construction so that it applies to this case, without the assumption that M is a group. In order to make the construction iterable, we do the principal and associated constructions together. (An alternate noniterative construction is possible, and it gives spaces of the same weak homotopy type as the construction we are about to exhibit. We give further details in an appendix.)

We have the same data as before: the right M-space (X, μ), the principal quasifibration $r: C \to A$, the equivariant map $\phi: C \to X$. In addition, we have a left M-space (F, η), a quasifibration $q: D \to A$ with fibre F, and an "evaluation" map $\varepsilon: C \times F \to D$ such that

a) ε preserves fibres, that is, $q(cf) = r(c)$,

b) $\varepsilon(c, \ ): F \to q^{-1}(r(c))$ is a weak homotopy equivalence. (Think of C as Prin D.)

We imbed q in a quasifibration $\hat{q}: \hat{D} \to \hat{A}$, where $\hat{A}$ is as before, and where $\hat{D}$, except for the use of a stronger topology, is given by the expression

$$\hat{D} = D \underset{\varepsilon}{\cup} C \times I \times F \underset{\phi \times 1}{\cup} X \times F.$$

The topology on $\hat{D}$ is the strongest topology with respect to which the coordinate functions

$$t: \hat{D} \to [0, 1], \qquad c: t^{-1}(0, 1) \to C, \qquad f: t^{-1}[0, 1) \to F,$$

$$cf: t^{-1}(0, 1] \to \dot{D}, \qquad \phi(c): t^{-1}[0, 1) \to X$$

are continuous.

The principal construction $\hat{r}: \hat{C} \to \hat{A}$ is also defined as before. We define an extension $\hat{\varepsilon}: \hat{C} \times F \to \hat{D}$ of ε by $\hat{\varepsilon}(c\,|t|\,m, f) = c\,|t|\,mf$.

THEOREM 4.1. *If* (F, η) *is a weak representation of* M, *then* $\hat{q}$ *is a quasifibration.*

The construction can be iterated, and we pass to the limit as before. If we start with $q: X \times F \to X$ and $r: X \times M \to X$ by projection with $\varepsilon(x, m, f) = (x, mf)$, we denote the limit q_∞ by $\tilde{q}: X \tilde{\times}_M F \to X_M$. The main properties of $\tilde{q}$ (to be proved) will explain and justify the notation.

That the base space is precisely X_M can be seen directly from the definition. Alternatively, X_M can now be described as $X \tilde{\times}_M *$, where $*$ is a point.

On the other hand, if we replace X by $*$, we get a map $X \tilde{\times}_M F \to * \tilde{\times}_M F$. We might call the latter space ${}_M F$ to emphasize the sidedness of the operation, but we continue to use F_M for both left and right M-spaces. If we replace both X and F by $*$, we get precisely the universal base space B_M of Dold and Lashof. Since maps of any M-space into a point are equivariant, we obtain the diagram

$$\begin{array}{ccc} X \tilde{\times}_M F & \longrightarrow & F_M \\ \downarrow & & \downarrow \\ X_M & \longrightarrow & B_M \end{array}$$

of quasifibrations with fibre F.

Corresponding fibres are mapped homeomorphically. The corresponding diagram for G-bundles exhibits the classifying map for $X \times_G F \to X_G$ in terms of the universal example $\mathcal{E}_G \times_G F \to B_G$. For fibre spaces, a universal example is known [13] if M is the monoid H(F) of all homotopy equivalences of F.

THEOREM 4.2. *The quasifibration* $F_{H(F)} \to B_{H(F)}$ *is a universal example of a quasifibration with fibre* F.

Proof. The universal example $u: UE \to B_{H(F)}$ is obtained from a quasifibration $\text{Ult}(\theta): \text{Ult}(F) \to B_{H(F)}$ by making it into a Hurewicz fibration in the standard way; but $\text{Ult}(F) \to B_{H(F)}$ is precisely $F_{H(F)} \to B_{H(F)}$, except for the topologies involved. The homotopy equivalence necessary to prove the theorem is discussed in the appendix.

5. THE ASSOCIATION BETWEEN PRINCIPAL QUASIFIBRATIONS AND QUASIFIBRATIONS WITH FIBRE F

The previous construction associates a quasifibration with fibre F with a principal quasifibration over M if F is a left M-space. For $M = \mathscr{H}(F)$, we can reverse the process by associating with $q: D \to A$ the map $\text{Prin}\ q: \text{Prin}\ D \to A$ defined by

$$\text{Prin}\ D = \{\phi: F \to q^{-1}(a) \mid \phi \text{ is a weak homotopy equivalence}\}.$$

We use the compact-open topology and assume henceforth that F is locally compact. Unfortunately, it is not known whether Prin q is again a quasifibration. However, if q is a Hurewicz (Covering Homotopy Property) fibration, then Prin q has the same property [13], and every quasifibration is quasi-equivalent to a Hurewicz fibration.

THEOREM 5.1 (compare Theorem B). *If* $r\colon X \to A$ *is a principal quasifibration over* $\mathscr{H}(F)$, *then* $\mathrm{Prin}\,(X \times_{\mathscr{H}(F)} F) \to X_{\mathscr{H}(F)}$ *is structurally equivalent to* $r\colon X \to A$.

This follows from a corresponding relation between the basic constructions.

LEMMA 5.2. *In terms of the basic constructions, if* $r = \mathrm{Prin}\, q$ *is a principal quasifibration, then so is* $\mathrm{Prin}\, \hat{q}$, *and it is structurally equivalent to* $\hat{r}$.

Proof. We see directly that if $M = \mathscr{H}(F)$, then

$$\mathrm{Prin}\,(D \cup C \times I \times F \cup X \times F) \;=\; \mathrm{Prin}\, D \cup C \times I \times M \cup X \times M$$

as sets. The map $\widehat{\mathrm{Prin}\, D} \to \mathrm{Prin}\, \hat{D}$ is determined by $(c\,|t|\,\phi)(f) = c\,|t|\,\phi(f)$. If F is locally compact, this map is continuous, because the adjoint $\mathrm{Prin}\, D \times F \to \hat{D}$ given by $(c\,|t|\,\phi, f) \to c\,|t|\,\phi(f)$ is continuous. Since the two spaces agree on $\mathrm{Prin}\, D = \hat{r}^{-1}(A)$, some fibres are mapped homeomorphically; from this the lemma follows.

Iterating application of the lemma, we see that $\mathrm{Prin}\,(X \tilde{\times}_M F) \to X_M$ is structurally equivalent to $\tilde{X} \to X_M$, which we have shown to be structurally equivalent to $\hat{r}$.

THEOREM 5.3 (compare Theorem A). *A quasifibration* $q\colon D \to A$ *with fibre* F *is quasi-equivalent to* $\mathrm{Prin}\, D \tilde{\times}_{\mathscr{H}(F)} F \to (\mathrm{Prin}\, D)_{\mathscr{H}(F)}$ *if* Prin q *is also a quasifibration.*

Proof. $X \tilde{\times}_M F$ is obtained by iterating the basic associated construction on $X \times F \to X$. Using the evaluation map $\varepsilon\colon \mathrm{Prin}\, D \times F \to D$, we can also apply the iterated basic construction to $q\colon D \to A$. Since

$$\begin{array}{ccc} \mathrm{Prin}\, D \times F & \longrightarrow & D \\ \downarrow & & \downarrow \\ \mathrm{Prin}\, D & \longrightarrow & A \end{array}$$

is commutative, we obtain, for $X = \mathrm{Prin}\, D$, the diagram

$$\begin{array}{ccc} X \tilde{\times}_{\mathscr{H}(F)} F & \longrightarrow & D_\infty \\ \downarrow & & \downarrow \\ X_{\mathscr{H}(F)} & \longrightarrow & A_\infty \end{array}$$

Since fibres are mapped by weak homotopy equivalences and since $X_{\mathscr{H}(F)} \to A_\infty$ is a weak equivalence (provided Prin q is a quasifibration), it is enough to show that

$$\begin{array}{ccc} D & \longrightarrow & D_\infty \\ \downarrow & & \downarrow \\ A & \longrightarrow & A_\infty \end{array}$$

gives a quasi-equivalence. Indeed, both bases and fibres are mapped by weak homotopy equivalences.

In summary, we have exhibited an equivalence between principal quasifibrations over $\mathscr{H}(F)$ and quasifibrations with fibre F, much as for bundles. Quasifibrations with structural monoid M and fibre F (where M is represented by weak homotopy equivalences of F) can now be defined in terms of our construction.

Definition 5.4. $q\colon D \to A$ is a *quasifibration with fibre* F *and structure monoid* M if there exists a principal quasifibration $r\colon X \to A$ over M such that $X \tilde{\times}_M F \to X_M$ is quasi-equivalent to q.

For weak fibrations, that is, for the case where $q\colon D \to A$ is locally fibre-homotopy trivial, this notion can be investigated via transition functions g_{ij}, as in the bundle case. The relation $g_{ij} = g_{ik} g_{kj}$ is no longer valid, but must be replaced by a suitable homotopy condition. This has been done successfully by Wirth [16].

To complete the present study of our construction, we show how this construction is compatible with the classic construction for groups and bundles.

6. SPECIALIZATION TO GROUPS AND BUNDLES

Dold and Lashof have shown that if G is a topological group and $p\colon E \to B$ is a principal G-bundle, then $p_\infty\colon E_\infty \to B_\infty$ is again a principal G-bundle. In order to be able to verify this result, they used the strong topology. The same proof gives the following theorem.

THEOREM 6.1. *If* G *is a topological group and* $r\colon X \to A$ *is a principal* G-*bundle, then* $r_\infty\colon X_\infty \to A_\infty$ *is a principal* G-*bundle.*

COROLLARY 6.2. *The principal* G-*bundle* r *is principally equivalent to the pull-back of* r_∞.

In fact, r is just the part of r_∞ lying over $A \subset A_\infty$.

THEOREM 6.3 (compare Theorem C). *If* $r\colon X \to A$ *is a principal* G-*bundle, where* G *acts as a group of homeomorphisms of* F, *then* $X \tilde{\times}_G F \to X_G$ *is a* G-*bundle, and it is* G-*equivalent to the pull-back of* $X \times_G F$ *over* $X_G \to A_\infty \to A$.

Proof. It is sufficient to consider the case $F = G$, with G acting by translation, because $(X \tilde{\times}_G G) \times_G F$ is homeomorphic to $X \tilde{\times}_G F$. Again, the proof given by Dold and Lashof shows that $X \tilde{\times}_G G \to X_G$ is a principal G-bundle. Looking once more at the map

$$\begin{array}{ccc} X \tilde{\times}_G G & \longrightarrow & X_\infty \\ \downarrow & & \downarrow \\ X_G & \longrightarrow & A_\infty \end{array},$$

we see that it is a map of principal G-bundles covering the weak homotopy equivalence $X_G \to A_\infty$. We see that fibres are mapped homeomorphically by looking at the construction and at the map

$$\begin{array}{ccc} X \times G & \xrightarrow{\mu} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle r} \\ X & \xrightarrow[r]{} & A \end{array}$$

of principal G-bundles. Since $r = r_\infty \mid X$ and since A is a deformation retract of A_∞, it follows that $X \tilde{\times}_G G \to X_G$ is the pull-back of r over $X_G \to A_\infty \to A$. Now, applying the operation $\times_G F$, we get the desired result.

Finally we compare our construction of X_G with that of Borel for an arbitrary antirepresentation of G by homeomorphisms of X. We notice first that if G is a topological group, then $\tilde{X} \to X_G$ is given by identification under the action of G, that is, $\tilde{X}/G = X_G$.

THEOREM 6.4. *$\tilde{X}/G$ has the same weak-homotopy type as Borel's space* $X_G = X \times_G \mathcal{E}_G$.

Proof. The maps $\phi: \tilde{X} \to X$ and $X \to *$ are equivariant; hence they induce an equivariant map $\tilde{X} \to X \times \mathcal{E}_G$ and thus a principal map

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \times \mathcal{E}_G \\ \downarrow & & \downarrow \\ \tilde{X}/G & \longrightarrow & X \times_G \mathcal{E}_G \end{array}$$

Since $X \subset \tilde{X}$ is a weak homotopy equivalence, since ϕ is a retraction, and since $\mathcal{E}_G$ has the weak homotopy type of a point, the map $\tilde{X} \to X \times \mathcal{E}_G$ is a weak homotopy equivalence. Therefore the map $\tilde{X}/G \to X \times_G \mathcal{E}_G$ is a weak homotopy equivalence.

Our extension of the concepts represented by X_G and $X \tilde{\times}_G F$ is compatible with the original concepts, up to weak homotopy type.

7. APPENDIX. AN ALTERNATE DESCRIPTION

The Dold and Lashof construction has been reworked, reformulated, retopologized, and even generalized by several people [7], [11], [14], [15]. It is itself a reworking and generalization of Milnor's construction [8]. Here we give one particularly simple reworking of our construction $X \tilde{\times}_M F$ that permits a direct rather than inductive definition. It is closest in form to Milgram's geometric bar construction [7]. The topology is not the strong topology, so we do not obtain the full strength of some of our results this way, but we emphasize the relation to homological algebra.

As before, we consider a principal quasifibration $r: X \to A$ over M and a weak representation of M on F. Let Δ^n denote the standard n-simplex; barycentric coordinates will be used. In $\Delta^n \times X \times M^n \times F$, consider the equivalence relation given by

$$(t_0, \cdots, t_n, x, m_1, \cdots, m_n, f) \sim (t_0, \cdots, t_n, x', m_1', \cdots, m_n', f')$$

if $t_i = 0$ and $m_i m_{i+1} = m_i' m_{i+1}'$, where $m_0 = x$ and $m_{n+1} = f$. Let D_n be the quotient space. Let A_n be obtained by replacing F by * throughout, and let $q_n: D_n \to A_n$ be induced by $F \to *$. An embedding of q_n in q_{n+1} is induced by

$$(t_0, \cdots, t_n, x, m_1, \cdots, m_n, f) \to (t_0, \cdots, t_n, 0, x, m_1, \cdots, m_n, e, f),$$

where $e \in M$ is the unit. Let $q_\infty: D_\infty \to A_\infty$ be the limit under these inclusions.

THEOREM 7.1. $q_\infty: D_\infty \to A_\infty$ *is quasi-equivalent to* $X \tilde{\times}_M F \to X_M$.

Proof. First we note that D_∞ and $X \tilde{\times}_M F$ are isomorphic as sets. The inductive step is the observation that D_{n+1} has the same underlying set as $\hat{D}_n = D_n \cup C_n \times I \times F \cup X \times F$. Here C_n denotes the associated principal quasifibration over M, which can be regarded as obtained from the above construction with $F = M$. First, we show that C_{n+1} has the same underlying set as

$\hat{C}_n = C_n \cup C_n \times I \times M \cup X \times M$. A specific correspondence $\psi\colon C_{n+1} \to \hat{C}_n$ is defined by

$$\psi(t_0, \cdots, t_{n+1}, x, m_1, \cdots, m_{n+1}) = (xm_1 \cdots m_n, m_{n+1}) \quad \text{if } t_{n+1} = 1,$$

$$= (t'_0, \cdots, t'_n, x, m_1, \cdots, m_n)|t_{n+1}|\, m_{n+1}$$

$$\text{if } t_{n+1} < 1, \text{with } t'_i = t_i/t_{n+1}.$$

PROPOSITION 7.7. *ψ is a homotopy equivalence.*

Proof. We are confronted with the difference in topologies. The map ψ from the weak to the strong topologies is continuous. The topologies agree on the compact sets of C_{n+1} but this is *not* enough to establish even a weak homotopy equivalence (as has sometimes been stated). We construct a simple deformation of the identity to a map ζ that will be continuous as a map $\hat{C}_n \to C_{n+1}$. The idea is essentially that of Milnor [9], and it is illustrated by the simple example of the two topologies on SX, where $X = (0, 1)$.

Let $h_s\colon I \to I$ be a deformation that shrinks $[0, 1/4]$ to 0 and $[3/4, 1]$ to 1. Let $\zeta\colon \hat{C}_n \to C_{n+1}$ be given by

$$c\,|t|\, m \to \psi^{-1}(c\,|h_1(t)|\, m).$$

One verifies directly that ζ is continuous. The homotopies $\psi\zeta \simeq \text{id}$ and $\zeta\psi \simeq \text{id}$ are easy to write down.

The same method shows that A_n has the same homotopy type as the appropriate iteration of our construction in the strong topology.

These homotopy equivalences also show that q_n is a quasifibration; hence, by a standard lemma for quasifibrations, the limit q_∞ is also a quasifibration. To verify the quasi-equivalence between q_∞ and $\tilde{X} \to X_M$, we need only remark that in both cases, the limit topology is used. The same arguments can now be applied for $X \tilde{\times}_M F \to X_M$.

The noniterative description given here reveals the relation to homological algebra fairly clearly. A preliminary deformation will permit even greater clarity.

Suppose in our iterative construction we "reduce," as in forming the reduced cone or reduced join. That is, defining D_n, we further identify

$$(t_0, \cdots, t_n, x, m_1, \cdots, m_n, f)$$

with $(t'_0, \cdots, t'_n, x, m_1, \cdots, m_n, f)$ if $m_i = e$ and $t_{i-1} + t_i = t'_{i-1} + t'_i$. That this is indeed a deformation can be seen by induction, looking at D_{n+1} as

$$D_n \cup C_n \times I \times F \cup X \times F.$$

With our usual representation of points in these spaces, the further identification corresponds to identifying $(c\,|s|\, e)|t|\, f$ with $(c\,|s'|\, e)|t'|\, f$ if $s + t = s' + t'$. In $C_{n-1} \times I \times M \times I \times F$, this amounts to shrinking line segments given by $s + t = \text{constant}$.

Let us assume that $X \tilde{\times}_M F$ has been reconstructed in this way. We can regard it as filtered by the reconstructed subspaces D_n. For each generalized cohomology theory T^*, we obtain a spectral sequence with $E_1 = \bigoplus_p T^*(D_p, D_{p-1})$. If T^q (point) is trivial for sufficiently large q, the spectral sequence converges to the associated graded group of $T^*(X \tilde{\times}_M F)$. Under suitable restrictions (for example, with field coefficients in singular cohomology), E_1^p can further be identified with

$$T^*(X) \otimes T^*(M, e) \otimes \cdots \otimes T^*(M, e) \otimes T^*(F).$$

If $X = M$ and F is a point, E_1 is recognizable as a standard free acyclic resolution of $T^*(M)$ as a *co*-algebra. For general X and F, this enables us to identify E_2.

THEOREM 7.3. *If* $T^*(D_p, D_{p-1})$ *is naturally isomorphic with*

$$T^*(X) \otimes T^*(M, e) \otimes \cdots \otimes T^*(M, e) \otimes T^*(F),$$

then

$$E_2 \approx \mathrm{Coext}_{T^*(M)}(T^*(X), T^*(F)).$$

In this way, we regard our construction as giving a geometric realization and generalization of the Eilenberg-Moore spectral sequence for $X \times_G F$.

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University of Notre Dame
Notre Dame, Indiana 46556