THE SECTION EXTENSION THEOREM AND LOOP FIBRATIONS

Martin Fuchs

In the discussion of principal fibrations, one has to spend some time on partitions of unity in the theory of principal fibrations. Thus most of this paper originates from [2]. Because equivariant fibrations have recently attracted much interest, we have tried to use their language as a vehicle. Loop fibrations provide an easy application of this theory, and we can generalize and correct a result of [1].

Let H_0 be a strictly associative H-space with strict unit element ε. Except where we state the contrary, all discussions in this paper are restricted to the category $\mathfrak{C}_0$ of topological spaces, with an operation of H_0 and with continuous maps that are compatible with the operations involved. All operations are assumed to be associative, with ε acting as the identity map.

If X is an object in $\mathfrak{C}_0$, the action of H_0 on X is in general not a morphism in $\mathfrak{C}_0$; in particular, the multiplication of H_0 is an action on H_0 but not a morphism in $\mathfrak{C}_0$. When we refer to H_0 as object in $\mathfrak{C}_0$, we refer to this action on H_0.

The unit interval I in this category is [0, 1], together with the trivial operation $h(x) = x$ for all $h \in H_0$ and $x \in [0, 1]$. If X and Y are in $\mathfrak{C}_0$, the product of X and Y is $X \times Y$, together with the diagonal action. Thus we can use haloing functions, halos [2, Definition 2.1], and homotopies in $\mathfrak{C}_0$ (haloing functions are obviously constant on orbits).

Following a suggestion of D. Puppe, we consider the following reformulation of the section extension property.

Let $E \xrightarrow{p} B$ be a map onto B in $\mathfrak{C}_0$; then a cross-section $s: B \to E$ is a map such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{s} & E \\ & {\scriptstyle 1_B}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

is commutative. If $E' \xrightarrow{p'} B'$ is also a map onto B' in $\mathfrak{C}_0$, we consider the two diagrams

$$(1) \qquad \begin{array}{ccc} E & \overset{\sigma}{\dashrightarrow} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}, \qquad\qquad (2) \qquad \begin{array}{ccc} E & \overset{\sigma}{\dashrightarrow} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \overset{\bar{\sigma}}{\dashrightarrow} & B' \end{array}$$

We have two problems:

Received December 27, 1967.
The author acknowledges support from the National Science Foundation.

α) Given f: B → B', to find σ: E → E' such that (1) commutes.

β) To find $\bar{\sigma}$: B → B' and σ: E → E' such that (2) commutes.

To facilitate references to Dold's paper, we call each of the pairs (σ, f) and $(\sigma, \bar{\sigma})$ a section S from B to p' or from p to p'.

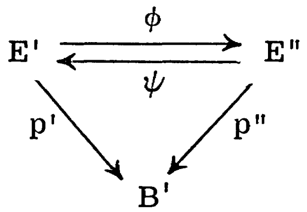
Definition. A space p: E → B over B has the *section extension property* (SEP) with respect to p': E → B if for every $A \subset B$ and every section S from A to p' that admits an extension to a halo V around A, there exists an extension S from B to p' (the inclusion map is of course to be interpreted in the category $\mathfrak{C}_0$). If S is of the form $(\sigma, \bar{\sigma})$, we speak of ASEP (absolute section extension property); if $S = (\sigma, f)$, we speak of RSEP (relative section extension property). SEP refers to both, ASEP and RSEP.

PROPOSITION. *Let* p: E → B *and* p': E' → B' *be spaces over* B *and* B', *respectively.*

α) *If* p': E' → B' *is dominated by* p": E" → B' (see [2]), *and if* p *has* SEP *with respect to* p", *then* p *has* SEP *with respect to* p'.

β) *If* p: E → B *is dominated by* p": E" → B, *and if* p" *has* SEP *with respect to* p', *then so does* p.

Proof (see Dold [2, Proposition 2.3]). α) I. The map p" dominates p'; that is, there exist maps ϕ and ψ such that the diagram



commutes, and we have a vertical homotopy $\theta : \psi\phi \simeq 1_{E'}$ over B' (with $\theta_0 = \psi\phi$, $\theta_1 = 1_{E'}$).

II. Let $A \subset V \subset B$, where V is a halo around A with respect to the haloing function τ: B → I, and let $S = (\sigma, \bar{\sigma})$ be a section from A to p' extendable to V, the extension being again called S. Then, in the case of ASEP, consider

$$\sigma' : E \,|\, V \xrightarrow{\sigma} E' \xrightarrow{\phi} E'' \quad (S = (\sigma, \bar{\sigma})) \qquad \text{and} \qquad \bar{\sigma}' = \bar{\sigma} : V \to B'.$$

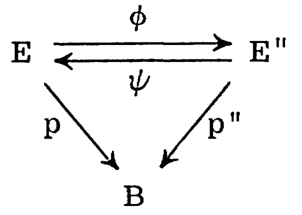
Then $(\sigma', \bar{\sigma}')$ has an extension to B, say S', such that S' restricted to $\tau^{-1}[1/2, 1]$ is the same as $(\sigma', \bar{\sigma}')$ restricted to $\tau^{-1}[1/2, 1]$. We denote S' again by $(\sigma', \bar{\sigma}')$.

Define $\bar{\sigma} = \bar{\sigma}'$ and

$$\sigma(y) = \begin{cases} \psi\sigma'(y) & \text{if } \tau p(y) \leq 1/2, \\ \theta(\sigma'(y), 2\tau p(y) - 1) & \text{if } \tau p(y) \geq 1/2. \end{cases}$$

The relative case is proved similarly.

β) I. The map p" dominates p; that is, there exist maps ϕ and ψ such that the diagram



commutes, and we have a vertical homotopy $\theta: \psi\phi \simeq 1_E$ over B (with $\theta_0 = \psi\phi$, $\theta_1 = 1_E$).

II. Let A, V, B, τ, and S satisfy the conditions in part α) of the proof. Then, in the case of ASEP, consider the pair $(S = (\sigma, \bar{\sigma}))$ with

$$\sigma': E'' \mid V \xrightarrow{\psi} E \xrightarrow{\sigma} E' \quad \text{and} \quad \bar{\sigma}' = \bar{\sigma}: V \to B'.$$

Then $(\sigma', \bar{\sigma}')$ has an extension to B (again, we write $S' = (\sigma', \bar{\sigma}')$) such that S' restricted to $\tau^{-1}[1/2, 1]$ is the same as the old $(\sigma', \bar{\sigma}')$ restricted to $\tau^{-1}[1/2, 1]$. Define $\bar{\sigma} = \bar{\sigma}'$ and

$$\sigma(y) = \begin{cases} \sigma'\phi(y) & \text{if } \tau p(y) \leq 1/2, \\ \sigma'(\theta(y, 2\tau p(y) - 1)) & \text{if } \tau p(y) \geq 1/2. \end{cases}$$

The relative case works similarly.

PROPOSITION (see Proposition 2.6 in [2]). *If* p: E → B *has the* SEP *with respect to* p', *and if* $W \subset B$ *is an open set such that* $W = \rho^{-1}(0, 1]$ *for some continuous function* $\rho: B \to [0, 1]$, *then* $p_W: E \mid W \to W$ *has the* SEP.

The proof, essentially an approximation process with haloing functions, is the same as in [2].

SECTION EXTENSION THEOREM (see [2, Theorem 2.7]). *Let* p: E → B *and* p': E' → B' *be spaces over* B *and* B', *respectively.*

α) *If there exists a numerable covering* $\{V_\lambda\}_{\lambda \in \Lambda}$ *of* B *such that* p *has the* ASEP *over each* V_λ, *with respect to* p', *then* p *has the* ASEP *with respect to* p'.

β) *Let* f: B → B' *be a map. If there exists a numerable covering* $\{V_\lambda\}_{\lambda \in \Lambda}$ *of* B *such that, for each* $\lambda \in \Lambda$, $p \mid p^{-1}(V_\lambda)$ *has the* RSEP *with respect to* p' *restricted to* $E' \mid f(V_\lambda)$, *then* p *has the* RSEP *with respect to* p'.

Again the proof is as in [2, Corollary 2.8].

COROLLARY 1. *Suppose that* p: E → B *and* p': E' → B' *are spaces over* B *and* B', *respectively, that* $A \subset B$, *and that* $S_A = (\sigma_A, \bar{\sigma}_A)$ *is a section from* A *to* p' *with an extension* $S_V = (\sigma_V, \bar{\sigma}_V)$ *to a halo* V *around* A. *If* f: B → B' *is an extension of* $\bar{\sigma}_V: V \to B'$ *and if* B - A *has a numerable covering* $\{V_\lambda\}_{\lambda \in \Lambda}$ *such that* $E' \mid fV_\lambda$ *is dominated by* fV_λ *(fiberwise), then there exists a section* S *from* B *to* p' *extending* S_A.

If $f = 1_B$, this is Corollary 2.8(α) of [2].

Definition. A fibration p: E → B in $\mathfrak{C}_0$ is a *principal fibration* if $\mu: E \times H_0 \to E$ is fiber-preserving, that is, H_0 acts trivially on B, and if moreover $\mu(y, \varepsilon) = y$ for all $y \in E$ (ε denotes the neutral element of H_0).

Example. $B \times H_0 \xrightarrow{pr_1} B$ is a principal fibration if H_0 acts trivially on B and if the action on H_0 is the multiplication of H_0.

COROLLARY 2. *Let* $p: E \to B$ *and* $p': E' \to B'$ *be principal fibrations such that* E' *is contractible in* $\mathfrak{T}$ *(the category of topological spaces and continuous maps). Suppose* $A \subset B$ *and* $S_A = (\sigma_A, \bar{\sigma}_A)$ *is a section from* A *to* p' *that has an extension* $S_V = (\sigma_V, \bar{\sigma}_V)$ *to a halo* V *around* A. *If* $B - A$ *has a numerable covering* $\{V_\lambda\}_{\lambda \in \Lambda}$ *such that* $p \mid p^{-1} V_\lambda$ *is dominated by* $p_\lambda = pr_1: V_\lambda \times H_0 \to V_\lambda$, *then there exists a section* S *from* B *to* p' *such that* $S \mid A = S_A$.

The proof of Corollaries 1 and 2 is the same application of the section extension theorem as in [2, Corollary 2.8(α)]. To see that $V_\lambda \times H_0 \xrightarrow{p_\lambda} V_\lambda$ has the ASEP with respect to p', let $A \subset V \subset V_\lambda$ be a halo around A in V_λ with haloing function τ. Let $k: E' \times I \to E'$ be a contraction of E', and let $S_V = (\sigma_V, \bar{\sigma}_V)$ be a section from V to p'. (Assume $k(y', 0) = y_0$, $k(y', 1) = y$.) Define $\bar{\sigma}$ and κ by

$$\bar{\sigma}(b) = p'\kappa(b) = \begin{cases} p' \circ k(\sigma(b, \varepsilon), 2\tau(b) - 1) & \text{if } 1/2 \leq \tau(b) \leq 1, \\ p' y_0' & \text{if } 0 \leq \tau(b) \leq 1/2, \end{cases}$$

and let $\sigma(b, h) = \mu'(\kappa(b), h)$, where $\mu': E' \times H_0 \to E'$ is the operation of H_0 on E'. Then $S \mid A = S_A$ if $S = (\sigma, \bar{\sigma})$.

The following definitions enable us to formulate Corollary 2 in an equivariant form.

Definition. An H-*space in* $\mathfrak{C}_0$ is an associative H-space H with strict unit element, together with an operation $\mu_H: H \times H_0 \to H$ such that the (right) multiplication μ of H is a morphism in $\mathfrak{C}_0$ and $\mu_H(\varepsilon, h) = \varepsilon$ for all $h \in H_0$ (that is, the neutral element of H is an orbit).

Definition. An H_0-*principal fibration with fiber* H *in* $\mathfrak{C}_0$ is a principal fibration $p: E \to B$ with respect to H, together with operations μ_H and μ_E of H_0 on H and E such that the operation of H on E and the multiplication of H are morphisms in $\mathfrak{C}_0$. We denote by $\mathfrak{P}_0$ the subcategory of $\mathfrak{C}_0$ whose objects are H_0-principal fibrations with fiber H and whose morphisms are maps compatible with the various actions.

Now consider the H space $H' = H \times H_0$ with multiplication

$$\mu' = (\mu \times \mu_0)(1_H \times T \times 1_{H_0}): (H \times H_0) \times (H \times H_0) \to H \times H_0.$$

We say that H' operates on E if the diagram

$$\begin{array}{ccccc} E \times H \times H_0 & \xrightarrow{1 \times \Delta} & E \times H \times H_0 \times H_0 & \xrightarrow{1 \times T \times 1} & E \times H_0 \times H \times H_0 \\ \downarrow \mu_E(\mu \times 1_{H_0}) & & & & \downarrow \mu_E \times \mu_H \\ E & & \xleftarrow{\quad\quad \mu \quad\quad} & & E \times H \end{array}$$

is commutative ($\mu: E \times H \to E$ denotes the principal action of H on E).

The associativity of μ_E and μ can be expressed as associative actions of $H \times \varepsilon_0$ and $\varepsilon \times H_0$. Associative action of all of H' on E is more than just the associative actions μ and μ_E. Using H' instead of H and H_0, one can describe

H_0-principal fibrations with fiber H as special fibrations in the category $\mathfrak{C}'$ of spaces and maps with H-space H'. We leave it to the reader to verify that our results still hold.

Application. Let Y be a path-connected topological space, and let $\Omega(Y, y_0)$ denote the space of Moore loops in Y based at y_0 (see [3, p. 284]). Since this H-space fulfills the requirements of the H-space H_0, we can apply our results in $\mathfrak{C}_0$ to the category $\mathfrak{C}_{\Omega Y}$. Let X be a topological space in $\mathfrak{C}_{\Omega Y}$ with a numerable covering of contractible sets $\mathfrak{U}$. We say that $p: E \to B$ has the *weak covering homotopy property* (WCHP) if for all X in $\mathfrak{C}_0$ and all maps $h: X \times I \to B$ and $k_0: X \times \{0\} \to E$ such that $p \circ k_0 = h \mid X \times \{0\}$, there exists a map $k: X \times I \to E$ such that $p \circ k = h$ and $k \mid X \times \{0\}$ is vertically homotopic to k_0. We call $p: E \to X$ a *loop fibration* if

1) p has the weak covering homotopy property with respect to $\mathfrak{C}_{\Omega Y}$,

2) p is a principal fibration in $\mathfrak{C}_{\Omega Y}$ (note that $\Omega(Y_0)$ acts trivially on X).

Remark. Condition 1) is actually not needed, since Theorem 6.4 of [2] does hold in this case.

In particular, if $U \in \mathfrak{U}$, then $p^{-1}U \simeq U \times \Omega(Y, y_0)$; that is, $p^{-1}U$ is fiber-homotopy equivalent to $U \times \Omega(Y, y_0)$ in $\mathfrak{C}_{\Omega Y}$. Consider the loop fibration $E(Y, y_0) \xrightarrow{p_Y} Y$ consisting of the Moore paths in Y based at y_0 (see [3, p. 284]). Since $E(Y, y_0)$ is contractible in the category $\mathfrak{T}$ of ordinary topological spaces, Corollary 2 to the section extension theorem applies to any loop fibration over X. Thus, if $p: E \to X$ is a loop fibration, there exists in $\mathfrak{C}_{\Omega Y}$ a fibermap (or cross-section) $(f, \bar{f})$ from p to p_Y. Let $p_f: E_f \to X$ be the loop fibration induced by $(f, \bar{f})$ from p_Y, and let $(\tilde{f}, 1_X)$ be the canonical fiber map from p to p_f. If $U \in \mathfrak{U}$, then there exists a fiber homotopy equivalence

$$\alpha: U \times \Omega(Y, y_0) \to p^{-1}U \quad \text{and} \quad \beta_f: p_f^{-1}U \to U \times \Omega Y,$$

since U is contractible in X and both fibrations p and p_f have the WCHP in $\mathfrak{C}_{\Omega Y}$. Consider the maps

$$U \times \Omega Y \xrightarrow{\alpha} p^{-1}U \xrightarrow{\tilde{f}} p_f^{-1}U \xrightarrow{\beta} U \times \Omega Y,$$

and let $q = \beta_f \circ \tilde{f} \mid p^{-1}U \circ \alpha$; since $q(x, \omega) = (x, (pr_2 \circ q(\varepsilon)) \circ \omega)$, q is a fiber-homotopy equivalence in $\mathfrak{C}_{\Omega Y}$ [the homotopy inverse of q is

$$(x, \omega) \to (x, (pr_2 \circ q(\varepsilon))^{-1} \circ \omega)].$$

Thus $\tilde{f} \mid p^{-1}U$ is a fiber-homotopy equivalence, and we can apply Theorem 3.3 of [2]. (Note that we can make the fibration $q: R \to E$ in Theorem 3.3 into a fibration in $\mathfrak{C}_{\Omega Y}$, by using the obvious actions. Then our section extension theorem applies as well: all we need is Lemma 3.4 of [2], as it is stated there for topological spaces.)

PROPOSITION. $(\tilde{f}, 1): E \to E_f$ *is a fiber-homotopy equivalence.*

Theorem 1 of [3] implies that if f, g: $X \to Y$ are homotopic, then E_f and E_g are fiber-homotopy equivalent. We obtain the converse by reapplying Corollary 2 to the usual map

$$\begin{array}{ccc} E_f \times \dot{I} & \xrightarrow{F, G \circ k} & E(Y, y_0) \\ \downarrow & & \downarrow \\ X \times \dot{I} & \xrightarrow[f,g]{} & Y \end{array}$$

where k: $E_f \times \{1\} \to E_g \times \{1\}$ is a fiber-homotopy equivalence, and where F and G are the canonical maps from E_f and E_g into $E(Y, y_0)$.

THEOREM. *Let* X *be a space in* $\mathfrak{C}_{\Omega Y}$ *with a numerable covering of contractible sets and with* ΩY *acting trivially on* X. *The fiber-homotopy equivalence classes of loop fibration over* X *are in one-to-one correspondence to the homotopy classes of maps from* X *into* Y.

This theorem corrects Theorem 8.2 in [1]. It obviously extends to equivariant loop fibrations. Comparison with Theorems 1 and 2 of [3] shows that in the case of fibrations over X induced from $E(Y, y_0)$, fiber-homotopy equivalence in $\mathfrak{C}_{\Omega Y}$ implies equivalence (by loop fiber maps) in the sense of [3, p. 285].

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Michigan State University
East Lansing, Michigan 48823