PARTS IN ANALYTIC POLYHEDRA

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1. INTRODUCTION

The nontrivial Gleason parts in the maximal ideal spaces of several large classes of function algebras have been shown to carry analytic structures. For example, Wermer [7] has shown that a nontrivial part of a Dirichlet algebra is the continuous one-to-one image of the open unit disc in the complex plane, all functions in the algebra being analytic on this disc in the obvious sense. Hoffman [6] obtained the same result for logmodular algebras. Although the general conjecture that a nontrivial part always carries analytic structure does not hold (see [2]), the class of algebras for which it is true is much larger than the cases (of essentially one complex dimension) mentioned above.

In this paper we examine the parts of algebras on analytic polyhedra in n-dimensional complex space $\mathfrak{C}^n$. In particular, we consider the function algebra that is the uniform closure on an analytic polyhedron of the functions holomorphic in a neighborhood of the polyhedron. As one would hope, the parts carry analytic structures on which the functions in the algebra are analytic. We show that the nontrivial parts are analytic subvarieties in relatively compact open sets in $\mathfrak{C}^n$. Moreover, any part containing an isolated point (that is, a point at which the subvariety describing the part is 0-dimensional) must reduce to the trivial part consisting of just the point itself. Actually, by appealing to Hoffman's characterization (in terms of analytic varieties) of points in the minimal boundary of analytic polyhedra [5], we can make a stronger statement: *an isolated point of a part must lie in the minimal boundary.* Viewed as a statement about the connectedness of parts, this says that a component of a nontrivial part cannot consist of a point. We would like to show that in general the parts of polyhedra are connected analytic varieties, but we have not been able to do this.

In Section 2, we include a brief review of the relevant definitions and terminology as well as a summary of the properties of analytic subvarieties that we use later. The main tool, which shows that connected analytic subvarieties are always contained in a single part with respect to the algebra of bounded holomorphic functions, is established in Section 3. The final section contains the applications to analytic polyhedra.

2. PARTS AND ANALYTIC SUBVARIETIES

If X is a compact Hausdorff space and A is a function algebra on X, then there exists a natural equivalence relation on the space M_A of maximal ideals of A. To define this equivalence relation, we identify the set of nonzero multiplicative linear

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functionals on A with M_A and realize A as a function algebra on M_A *via* the Gelfand representation.

2.1. *Definition.* Two points p and q in M_A are *(Gleason) equivalent* with respect to A (we write $p \sim q$ or $p \sim_A q$) if

$$\sup\{|f(p) - f(q)|: f \in A, \|f\| \leq 1\} < 2 \qquad (\|f\| = \sup_{M_A} |f|).$$

This defines an equivalence relation on M_A, and the equivalence classes are called *(Gleason) parts.* The fact that "$\sim$" is an equivalence relation was noted by Gleason in [3]. A proof of this fact is given in [8].

2.2. *Definition.* A point x in M_A is a *peak point* of A if there exists a function f in A for which x is the unique point at which f attains its maximum modulus. If M_A is metrizable, the set of peak points is called the *minimal boundary* of A (see [1]).

A peak point is always a trivial part; in other words, the equivalence class containing the point is a singleton. In fact, it is not difficult to show that whenever a pair of points has the property that only one of the points lies in the maximum-modulus set of a function in the algebra, then they lie in different parts.

2.3. *Definition.* Let U be open in $\mathfrak{C}^n$. An *analytic subvariety* V in U is a relatively closed subset V of U with the property that to each point p in V there correspond an open neighborhood W_p of p in U and functions $f_1, f_2, \cdots, f_k$, holomorphic in W_p, such that $W_p \cap V = \{q \in W_p: f_i(q) = 0 \text{ for } i = 1, 2, \cdots, k\}$.

2.4. *Definition.* An analytic subvariety V in U is *irreducible* if the relation $V = V_1 \cup V_2$ (where V_1 and V_2 denote analytic subvarieties in U) implies that either $V = V_1$ or $V = V_2$.

2.5. *Definition.* An analytic subvariety V in U is *irreducible at a point* p in V if there exists an open set U_p in $\mathfrak{C}^n$ such that the subvariety $V \cap U_p$ in U_p is irreducible.

2.6. *Facts about analytic subvarieties.* Let V be an analytic subvariety in a relatively compact open set U in $\mathfrak{C}^n$. Then for each point p in V there exists a neighborhood U_p in $\mathfrak{C}^n$ such that the subvariety $V_p = V \cap U_p$ in U_p has the following three properties.

2.6.1. $V_p = V_1 \cup V_2 \cup \cdots \cup V_j$, where each V_i is a subvariety in U_p, irreducible at p [4, p. 89].

2.6.2. If V_p is irreducible at p, there exists a connected open dense subset V' of V_p and an integer $k \leq n$ such that V' is a complex submanifold of U_p of dimension k. The set $V_0 = V_p - V'$ is called the set of *singular points* of V_p [4, p. 111].

2.6.3. If V_p is irreducible at p, there exists a coordinate system in $\mathfrak{C}^n$ such that for some connected open set U^k in $\mathfrak{C}^k$ the projection map $\pi: \mathfrak{C}^n \to \mathfrak{C}^k$ is an s-sheeted covering map of $V_p - V_0$ onto $U^k - \pi(V_0)$, with $\pi(p) \in U^k$ (p becomes the origin in $\mathfrak{C}^n$ with respect to the new coordinate system); see [4, pp. 98 and 114].

2.7. *Definition.* A continuous function f on V is *holomorphic at* p in V if f is holomorphic on the manifold of nonsingular points of each irreducible branch of V_p given by facts 2.6.1 and 2.6.2. The function f is *holomorphic on* V if f is holomorphic at each point of V.

2.8. *Facts about holomorphic functions on analytic subvarieties.*

2.8.1. Let V be irreducible at p, and let V_p, π, U^k, V_0 be as given by 2.6.2 and 2.6.3. Let h be a bounded holomorphic function on V_p. For each point a in $U^k - \pi(V_0)$, there exists a neighborhood U_a such that $\pi^{-1}(U_a) \cap V_p = U_1 \cup \cdots \cup U_s$, where π maps U_i homeomorphically onto U_a. Let

$$h_a^i = (h \mid U^i) \circ \pi^{-1} \qquad (1 \leq i \leq s).$$

Then h_a^i is holomorphic on U_a. If $S(X_1, \cdots, X_s)$ is a symmetric polynomial in $X_1, \cdots, X_s$, then $S(h_a^1, \cdots, h_a^s) = S_a$ is holomorphic in U_a. The function S defined by $S(a) = S_a(a)$ is a well-defined bounded holomorphic function on $U^k - \pi(V_0)$, and it extends uniquely to a bounded holomorphic function on U^k [4, p. 99].

2.8.2. Let D be open in $\mathfrak{C}^n$, and let $g_1, g_2, \cdots, g_r$ be holomorphic functions in D. Let V be the variety of common zeros of $g_1, g_2, \cdots, g_r$ in D. If p is an isolated point of V and Q is a compact subset of D containing p, then there exists a function g, holomorphic on an open neighborhood of Q, such that $g(p) = 1$ while g vanishes on $Q \cap (V - p)$ [5].

3. CONNECTED ANALYTIC VARIETIES

3.1. *Notation.* Let M be a complex analytic manifold or an analytic subvariety in an open set U in $\mathfrak{C}^n$. Let Q be a compact set in $\mathfrak{C}^n$. Let $BH(M)$ be the algebra of bounded holomorphic functions on M, $H(Q)$ the algebra of functions holomorphic in a neighborhood of Q, and $A(Q)$ the completion of $H(Q)$ in the sup-norm on Q.

To obtain the main tool mentioned in the introduction, we need a well-known and easily established lemma about connected analytic manifolds. We omit the proof (see [8]).

3.2. LEMMA. *If* M *is a connected, complex analytic manifold and* $BH(M)$ *separates points on* M, *then* $p \sim q$ *with respect to* $BH(M)$, *for all* $p, q \in M$.

3.3. THEOREM. *Let* V *be a connected analytic subvariety in a relatively compact open set* U *in* $\mathfrak{C}^n$. *Then* V *is imbedded in* $M_{BH(V)}$, *and all points of* V *lie in the same part of* $BH(V)$.

Proof. That V is imbedded in $M_{BH(V)}$ follows immediately from the fact that $BH(U)$ separates points of U and $BH(U) \mid V \subset BH(V)$. For the main part of the theorem, it suffices to look at V locally and to consider only the irreducible case. In other words, we claim that if for each point p in V and each irreducible component V_i of $V \cap U_p$ in some neighborhood U_p of p in $\mathfrak{C}^n$ the theorem is true for V_i, U_p, and $BH(V_i)$, then the theorem holds as stated.

To establish the claim, we note that if $q_1, q_2 \in V_i$ and $q_1 \sim q_2$ with respect to $BH(V_i)$, then certainly $q_1 \sim q_2$ with respect to $BH(V)$. Also, if $q_1 \in V_i$ and $q_2 \in V_j$, where V_i and V_j are distinct irreducible components of $V \cap U_p$, then the relations

$$q_1 \sim p \text{ with respect to } BH(V_i) \qquad \text{and} \qquad q_2 \sim p \text{ with respect to } BH(V_j)$$

imply that $q_1 \sim q_2$ with respect to $BH(V)$. Hence, each point p of V has a neighborhood that lies in the same part as p with respect to $BH(V)$. Since V is connected, this means that V lies in a single part. Henceforth, we assume that V is irreducible at p. We assume that coordinates in $\mathfrak{C}^n$ have been chosen according to 2.6.3, and we work locally with the set of singular points V_0, the connected manifold

$V - V_0$ of nonsingular points, and the s-sheeted covering map π: $\mathfrak{C}^n \to \mathfrak{C}^k$ of $V - V_0$ onto $U^k - \pi(V_0)$ given by 2.6.2 and 2.6.3. Since $V - V_0$ is a connected complex analytic manifold, Lemma 3.2 implies that all points of $V - V_0$ lie in the same part with respect to $BH(V - V_0)$, hence with respect to BH(V). It thus suffices to show that if $p \in V_0$, then p is equivalent to some point in $V - V_0$. (If q is any other point in V_0, then localization at q and repetition of the argument at q shows that q is equivalent to some point in $V - V_0$ also.) Suppose p is not in the same part as $V - V_0$. Then no point of $V - V_0$ is equivalent to p. For each b in $U^k - \pi(V_0)$, let $\pi^{-1}(b) = \{b_1, \cdots, b_s\}$. Fix $a \in U^k - \pi(V_0)$. Then $a_1, a_2, \cdots, a_s \in V - V_0$, and none of the a_i is equivalent to p. Hence there exist functions $\{f_m\}$ in BH(V) such that $\|f_m\|_V \leq 1$ for all m, $f_m(p) \to 1$, and $f_m(a_i) \to 0$ $(i = 1, 2, \cdots, s)$. Let

$$g_m(b) = \frac{1}{s} \sum_{i=1}^{s} f_m(b_i) \quad \text{for } b \in U^k - \pi(V_0).$$

Then, by 2.8.1, each g_m extends uniquely to a function in $BH(U^k)$. Also,

$$\|g_m\|_{U^k} = \|g_m\|_{U^k - \pi(V_0)} \leq \frac{1}{s} \sum_{i=1}^{s} \|f_m\|_V \leq 1.$$

But

$$g_m(\pi(p)) = \lim_{b \to \pi(p)} g_m(b) = \frac{1}{s} \sum_{i=1}^{s} \lim_{b_i \to p} f_m(b_i) = \frac{1}{s} \sum_{i=1}^{s} f_m(p) = f_m(p).$$

Thus

$$g_m(\pi(p)) \to 1 \quad \text{and} \quad g_m(a) = \frac{1}{s} \sum_{i=1}^{s} f_m(a_i) \to 0.$$

Therefore a is not equivalent to $\pi(p)$ with respect to $BH(U^k)$, contrary to Lemma 3.2. We conclude that p lies in the same part as $V - V_0$, and the theorem is established.

3.4. COROLLARY. *Let* Q *be a compact set in* $\mathfrak{C}^n$. *Let* p *be an isolated point of a part of* A(Q). *If* W *is an open set in* $\mathfrak{C}^n$ *and* V *is an analytic subvariety in* W *with* $p \in V \subset Q$, *then* p *is an isolated point of* V.

Proof. Choose an open relatively compact neighborhood R of p in $\mathfrak{C}^n$ such that $R \cap V$ is connected. Then $R \cap V$ is a connected, analytic subvariety in $R \cap W$. Hence, the theorem implies that if $q \in R \cap V$, then $q \sim p$ with respect to $BH(R \cap V)$. By hypothesis, there exists an open neighborhood S of p in $\mathfrak{C}^n$ such that if $p \neq q$ and $q \in S \cap Q$, then q is not equivalent to p with respect to A(Q). Let $T = R \cap S$. Then $T \cap V = \{p\}$, for if $q \in T \cap V$ and $q \neq p$, then $q \in R \cap V$ and $q \sim p$ with respect to $BH(R \cap V)$. Since $q \in S \cap V$ and $q \neq p$, q is not equivalent to p with respect to A(Q). Now $V \subset Q$ implies $R \cap V \subset Q$, so that

$$A(Q) \mid R \cap V \subset BH(R \cap V).$$

Thus we also conclude that p is not equivalent to q with respect to $BH(R \cap V)$, a contradiction. Hence $T \cap V = \{p\}$, and p is an isolated point of V.

4. ANALYTIC POLYHEDRA

We shall now apply the results of Section 3 to analytic polyhedra. Let Q be an analytic polyhedron in $\mathfrak{C}^n$, that is, let Q be a compact set in $\mathfrak{C}^n$, of the form

$$Q = \{z \in D: |f_i(z)| \leq 1, \ i = 1, 2, \cdots, k\},$$

where D is open in $\mathfrak{C}^n$ and $f_1, f_2, \cdots, f_k$ are holomorphic in D. Consider the algebra A(Q).

4.1. THEOREM. *If* G *is a part of* A(Q), *then* G *is an analytic subvariety in a relatively compact open set in* $\mathfrak{C}^n$.

Proof. Since the maximal ideal space of A(Q) coincides with Q, we need only examine the part containing an arbitrary point p in Q. First suppose that $|f_i(p)| < 1$ $(i = 1, 2, \cdots, k)$. Then $U = \{z \in D: |f_i(z)| < 1, \ i = 1, 2, \cdots, k\}$ is an open set in $\mathfrak{C}^n$, and it contains the part containing p, by the remarks following Definition 2.2. Now, since $U \subset Q$, Gleason-equivalence of points in U with respect to BH(U) implies equivalence with respect to A(Q). According to Theorem 3.3, each component of U lies in a part of A(Q). Thus G must be a component of U or a union of components of U — hence an analytic subvariety (in U).

If $|f_i(p)| = 1$ for some i, assume without loss of generality that $|f_i(p)| = 1$ $(i = 1, 2, \cdots, r)$ and $|f_i(p)| < 1$ $(i = r+1, \cdots, k)$. Let

$$U = \{z \in D: |f_i(z)| < 1, \ i = r+1, \cdots, k\} \qquad (\text{if } r = k, \text{ take } U = D),$$

$$V = \{z \in U: f_i(z) = f_i(p), \ i = 1, 2, \cdots, r\}.$$

Then $V \subset Q$ and no point of Q not in V lies in the same part as p; for if q is not in U, then q lies in the set of points of maximum modulus of some f_{i_0} $(i_0 > r)$. If q is in U but not in V, then q does not lie in the set of points of maximum modulus of one of the functions $g_1, \cdots, g_r$, where $g_i = 1 + \bar{f}_i(p) f_i$ $(i = 1, 2, \cdots, r)$. Applying the argument in the previous case to V, we again conclude that G is either a component of V or a union of components of V — in any case, an analytic subvariety (in U).

To establish our claims about isolated points of parts, we use Hoffman's characterization of the points in the minimal boundary of A(Q); it states that a point p in Q is in the minimal boundary if and only if no local analytic variety through p lies in Q and has positive dimension at p. The specific theorem that we wish to apply is the following:

4.2. THEOREM (Hoffman). *Let*

$$Q = \{z \in D: |f_j(z)| \leq 1, \ j = 1, 2, \cdots, k\}$$

be an analytic polyhedron in $\mathfrak{C}^n$. *Let* p *be a point in* Q, *and suppose (renumbering, if necessary) that* $|f_j(p)| = 1$ $(j = 1, 2, \cdots, r)$. *Suppose* p *is an isolated point of the analytic subvariety*

$$V = \{z \in D: f_j(z) = f_j(p), \ j = 1, 2, \cdots, r\}.$$

Then p *is a peak point for the algebra* A(Q).

The theorem is a consequence of 2.8.2 and of a lemma of Bishop which, under certain conditions on a set in $\mathfrak{C}^2$, allows the construction of a peaking function. For a proof of both Bishop's lemma and the theorem, see [5].

4.3. THEOREM. *Let*

$$Q = \{z \in D: |f_j(z)| \leq 1,\ j = 1, 2, \cdots, k\}$$

be an analytic polyhedron in $\mathfrak{C}^n$. *If a point* p *of* Q *is an isolated point of a part of* A(Q), *then* p *is a peak point of* A(Q).

Proof. Since p is a "local point part" of A(Q), $|f_j(p)| = 1$ for some j; for otherwise p would lie in an open relatively compact set Q^o in $\mathfrak{C}^n$. Therefore p is an isolated point of a part of $BH(Q^o)$. By Lemma 3.2, p is an isolated point of Q^o, which is impossible. Without loss of generality, we assume that $|f_j(p)| = 1$ for $j = 1, 2, \cdots, r$. Let V be the variety in D described in Theorem 4.2; that is, let

$$V = \{z \in D: f_j(z) = f_j(p),\ j = 1, 2, \cdots, r\}.$$

Let $U = \{z \in D: |f_j(z)| < 1,\ j = r+1, r+2, \cdots, k\}$. Then $V \cap U$ is an analytic subvariety in U with $p \in V \cap U \subset Q$. By Corollary 3.4, p is an isolated point of $V \cap U$; in other words, there exists an open neighborhood W of p in $\mathfrak{C}^n$ such that $W \cap V \cap U = \{p\}$. This clearly implies that p is an isolated point of V. Therefore, by Theorem 4.2, p is a peak point of the algebra A(Q).

4.4. COROLLARY. *For analytic polyhedra, an isolated point of a part is a part in itself.* (An alternate statement might say that a local point part is a global point part.)

Proof. By a previous observation, a peak point is a trivial part. Apply the theorem.

4.5. COROLLARY. *If* Q *is an analytic polyhedron in* $\mathfrak{C}^n$, *then the set of trivial parts and the set of peak points for* A(Q) *coincide.*

This follows immediately from Theorem 4.3.

The next corollary provides a partial solution to a problem posed at a conference on commutative Banach algebras at Dartmouth in 1960 [9, Problem 8, p. 457]:

4.6. *Problem.* Let A be a function algebra on M_A. Let K be closed in M_A, and let A_K be the uniform closure on K of $A \mid K$. Suppose p is an interior point of K and a peak point of A_K. Does it follow that p is a peak point of A?

We answer the problem for the case of analytic polyhedra:

4.7. COROLLARY. *Let* Q *be an analytic polyhedron in* $\mathfrak{C}^n$. *Let* K *be closed in* Q, *and let* A_K *be the uniform closure on* K *of* $A(Q) \mid K$. *If* p *is an interior point of* K *and a peak point of* A_K, *then* p *is a peak point of* A(Q).

Proof. Suppose that S is open in $\mathfrak{C}^n$ and that $S \cap Q = K^o$. We show that if W is any open set in $\mathfrak{C}^n$ and V is an analytic subvariety in W with $p \in V \subset Q$, then p is an isolated point of V. To this end, let S' be open in $\mathfrak{C}^n$, and suppose that $p \in S' \subset \bar{S}' \subset S$. Let $W' = W \cap S'$, $V' = V \cap S'$, and $Q' = Q \cap \bar{S}'$. Then Q' is a compact set in $\mathfrak{C}^n$, V' is an analytic subvariety in W', an open set in $\mathfrak{C}^n$, and $p \in V' \subset Q'$. Also, $\underline{Q'} \subset K^o$. Since p is a peak point of A_K, it is a trivial part of A_K. But $A(Q') \supset A(\overline{K^o}) \mid Q' \supset A_K \mid Q'$. Thus no point q in the neighborhood $K^o \cap S'$ of p in Q' different from p is equivalent to p with respect to A(Q'). These conditions allow us to deduce, with the aid of Corollary 3.4, that p is an isolated point of V'. But then p is an isolated point of V. To obtain Corollary 4.7, we proceed as in the proof of Theorem 4.3.

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