COMPLETENESS OF $\{A \sin nx + B \cos nx\}$ ON $[0, \pi]$

Robert Feinerman and D. J. Newman

The classical L^2-theory of Fourier series tells us that on $[0, \pi]$ the sines are complete while the cosines are not (unless we include $1 = \cos 0x$). It is natural to ask the completeness question about the family $\{A \sin nx + B \cos nx\}$ (A and B arbitrary complex numbers), and indeed, to generalize to the other L^p-spaces. The question is also interesting in the case $p = \infty$, if here we consider instead of L^∞ (which is not separable) its subspace C of functions continuous on $[0, \pi]$. In this paper we give the complete answers to these questions.

These answers are most simply expressed in terms of a slightly different notation. Observe that if $A/B = \pm i$, we are looking at the set $\{e^{inx}\}$ or $\{e^{-inx}\}$, and that in this case completeness holds in the strongest topology of all, namely in $C[0, \pi]$ (and even in $C[0, \tau]$ for any $\tau < 2\pi$). If $A/B \neq \pm i$, we can write

$$A \sin nx + B \cos nx = \pm\sqrt{A^2 + B^2} \sin\left(nx + \frac{\pi}{2}\alpha\right),$$

where $-1 \leq \Re\alpha \leq 1$. Also, since the replacement of x by $\pi - x$ shows that completeness for α is equivalent to that for $-\alpha$, we impose the further restriction $0 \leq \Re\alpha \leq 1$. In all that follows, we shall assume this, and we shall denote by S_α the set of functions $\left\{\sin\left(nx + \frac{\pi}{2}\alpha\right)\right\}$ $(n = 1, 2, \cdots)$; also, we abbreviate $L^p[0, \pi]$ to L^p.

THEOREM 1. I. S_α *is complete in* $L^1 \Leftrightarrow \Re\alpha \neq 1$.

II. *Let* $1 < p < \infty$. S_α *is complete in* $L^p \Leftrightarrow \Re\alpha \leq 1/p$.

III. S_α *is complete in* $C \Leftrightarrow \Re\alpha = 0,\ \alpha \neq 0$.

In the α-notation, the completeness set for L^p is a strip, while for C it is the imaginary axis excluding the origin. If we map back to the B/A notation, these sets are "lens shaped." For $1 < p < \infty$, the completeness set in the B/A-plane consists of the inside and boundary of the curve formed by two circular arcs, each passing through $\pm i$ and making the interior angle π/p with the imaginary axis. (When $p = 2$, this set becomes the closed unit disc.)

For L^1, the completeness set is the entire plane except for the points iy on the imaginary axis with $|y| > 1$.

For C, the set consists of the points iy on the imaginary axis with $0 < |y| \leq 1$.

A strange situation exists in the case where B/A is imaginary. Here our theorem tells us that $\{\sin nx + i\lambda \cos nx\}$ is complete in the strongest sense (in C) if $0 < \lambda \leq 1$, and that for $\lambda > 1$ the family is not even complete in L^1.

If instead of letting n range through the positive integers, we throw in also $n = 0$, then completeness is essentially universal.

THEOREM 2. *If* $\alpha \neq 0$, *the set of functions* $1 \cup S_\alpha$ *is complete in* C.

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Our major tool in the analysis of these questions is the following identity.

Let $0 < \Re\alpha < 2$, $n = 1, 2, \cdots$; then

$$\int_0^\pi \left(\tan\frac{x}{2}\right)^{1-\alpha} \sin\left(nx + \frac{\pi}{2}\alpha\right) dx = 0, \tag{1}$$

where the principal branch of the power is taken (the branch that equals 1 at $x = \pi/2$).

It is clear how this identity will lead to incompleteness results, since it produces explicit functions orthogonal to the set S_α. What is curious is how we can also use the identity to obtain the completeness results! Roughly, the logic is this: We shall find ourselves in a situation where our set has codimension at most 1. Thus there will be at most one function, up to scalar multiples, orthogonal to this set. The identity therefore supplies the *only* possible such function. Then, if this function turns out not to be in the right class, we can conclude that no *non*-zero function in the class is orthogonal to the set, and the set will have been proved complete.

The proof of (1) is based on contour integration. By direct estimates, the integral

$$\int_{|z|=1} \left(\frac{1-z}{1+z}\right)^{1-\alpha} z^{n-1} dz$$

has the value

$$\lim_{r \to 1^-} \int_{|z|=r} \left(\frac{1-z}{1+z}\right)^{1-\alpha} z^{n-1} dz = 0.$$

On the other hand, if we set $z = e^{ix}$, the integral becomes

$$i \int_0^\pi \left(-i \tan\frac{x}{2}\right)^{1-\alpha} e^{inx} dx + i \int_{-\pi}^0 \left(-i \tan\frac{x}{2}\right)^{1-\alpha} e^{inx} dx$$

$$= \int_0^\pi \left(\tan\frac{x}{2}\right)^{1-\alpha} e^{\frac{\pi}{2}i\alpha} e^{inx} dx - \int_0^\pi \left(\tan\frac{x}{2}\right)^{1-\alpha} e^{-\frac{\pi}{2}i\alpha} e^{-inx} dx$$

$$= 2i \int_0^\pi \left(\tan\frac{x}{2}\right)^{1-\alpha} \sin\left(nx + \frac{\pi}{2}\alpha\right) dx,$$

and this gives (1).

We shall first prove Theorem 2, and that will furnish the information on codimension necessary in the proof of Theorem 1. The following lemma will supply the codimension information necessary in the proof of Theorem 2.

LEMMA. *If* $0 \le \Re\alpha \le 1/2$, *then* S_α *is complete in* L^2. *For any* α, *the set* $1 \cup S_\alpha$ *is complete in* L^2.

Proof. Let M denote the closed linear subspace of L^2 generated by the functions S_α, and suppose that $f(x) \in M^\perp$. If a_n and b_n denote the Fourier coefficients of $f(x)$, that is, if

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx \qquad (n = 0, 1, 2, \cdots),$$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx \qquad (n = 1, 2, \cdots),$$

then

(2) $$a_n \sin\frac{\pi}{2}\alpha + b_n \cos\frac{\pi}{2}\alpha = 0 \qquad \text{for } n = 1, 2, \cdots.$$

Also, by Parseval's theorem,

(3) $$\frac{2}{\pi}\int_0^\pi |f(x)|^2\,dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2,$$
$$\frac{2}{\pi}\int_0^\pi |f(x)|^2\,dx = \sum_{n=1}^{\infty} |b_n|^2.$$

Combining (2) and (3), we find that

(4) $$\frac{|a_0|^2}{2}\left|\sin^2\frac{\pi}{2}\alpha\right| = \left(\left|\sin^2\frac{\pi}{2}\alpha\right| - \left|\cos^2\frac{\pi}{2}\alpha\right|\right)\sum_{n=1}^{\infty}|b_n|^2.$$

Case 1. $1/2 < \Re\alpha \leq 1$. Here $\left|\sin\frac{\pi}{2}\alpha\right| > \left|\cos\frac{\pi}{2}\alpha\right|$, and so, by (4), we have the implication $a_0 = 0 \Rightarrow b_n \equiv 0$; that is, $a_0 = 0 \Rightarrow f(x) = 0$ a. e., and this means precisely that $1 \cup S_\alpha$ spans all of L^2.

Case 2. $0 \leq \Re\alpha < 1/2$. Here $\left|\sin\frac{\pi}{2}\alpha\right| < \left|\cos\frac{\pi}{2}\alpha\right|$, and therefore (4) ensures that $\sum_{n=1}^{\infty}|b_n|^2 \leq 0$ or (by (3)), that $f(x) = 0$ a. e. Thus

$$f(x) \in M^\perp \Rightarrow f(x) = 0 \text{ a. e.},$$

and we conclude that $M = L^2$.

Case 3. $\Re\alpha = 1/2$. Here $\left|\sin\frac{\pi}{2}\alpha\right| = \left|\cos\frac{\pi}{2}\alpha\right| \neq 0$, and therefore (4) implies that $a_0 = 0$. Thus $f(x) \in M^\perp \Rightarrow (f(x), 1) = 0$, and this means that

(5) $$1 \in M.$$

From (5) and the identity

$$2\cos x \cdot \sin\left(nx + \frac{\pi}{2}\alpha\right) = \sin\left((n+1)x + \frac{\pi}{2}\alpha\right) + \sin\left((n-1)x + \frac{\pi}{2}\alpha\right)$$

we conclude that

(6) $$\phi(x) \in M \Rightarrow \cos x \cdot \phi(x) \in M.$$

Starting with the function $\phi(x) = 1$, we deduce from (5) and repeated use of (6) that $1, \cos x, \cos 2x, \cos 3x, \cdots$ all lie in M. Since these functions form a complete family, we conclude again that $M = L^2$, and the lemma is proved.

Proof of Theorem 2. Choose any $\mu(x)$ of bounded variation on $[0, \pi]$, normalized so that $\mu(0) = 0$ and $\mu(x) \equiv (\mu(x^-) + \mu(x^+))/2$. Suppose that

$$\int_0^\pi d\mu(x) = 0, \quad \int_0^\pi \sin\left(nx + \frac{\pi}{2}\alpha\right) d\mu(x) = 0. \tag{7}$$

Then $\mu(\pi) = 0$, and integration by parts shows that

$$\int_0^\pi \mu(x)\cos\left(nx + \frac{\pi}{2}\alpha\right) dx = 0 \qquad (n = 1, 2, \cdots)$$

or, in more convenient form, that

$$\int_0^\pi \mu(\pi - x)\sin\left(nx + \frac{\pi}{2}(1 - \alpha)\right) dx = 0 \qquad (n = 1, 2, \cdots). \tag{8}$$

Case 1. $1/2 \leq \Re\alpha \leq 1$. By the lemma, the family $S_{1-\alpha}$ is complete in L^2. Since $\mu(\pi - x) \in L^2$, it follows from (8) that $\mu(\pi - x) = 0$ a.e., and so $\mu(\pi - x) = 0$ identically. Since $d\mu$ is any measure satisfying (7), we deduce the required completeness.

Case 2. $0 \leq \Re\alpha < 1/2$. This time our lemma only tells us that the set $S_{1-\alpha}$ has codimension at most 1 in L^2. This codimension is indeed 1, since our identity (1) implies that

$$\int_0^\pi \left(\tan\frac{x}{2}\right)^\alpha \sin\left(nx + \frac{\pi}{2}(1 - \alpha)\right) dx = 0 \qquad (n = 1, 2, \cdots), \tag{9}$$

and since $(\tan x)^\alpha \in L^2$. But since the codimension is 1, a comparison of (8) and (9) tells us that

$$\mu(\pi - x) = C\left(\tan\frac{x}{2}\right)^\alpha \tag{10}$$

almost everywhere (and hence everywhere).

By our assumption that $\alpha \neq 0$, the function $\left(\tan\frac{x}{2}\right)^\alpha$ is *not* of bounded variation on $[0, \pi]$. We are forced to conclude that $C = 0$ in (10), and so we deduce that $\mu(\pi - x) = 0$. Thus the completeness follows in this case also.

Proof of Theorem 1.

Proof of I, $\Rightarrow$. If $\Re\alpha = 1$, then $\left(\tan\frac{x}{2}\right)^{1-\alpha}$ is a bounded function, and so (1) shows that S_α is not complete.

Proof of II, $\Rightarrow$. If $1 \geq \Re\alpha > 1/p$, then $\left(\tan\frac{x}{2}\right)^{1-\alpha}$ is a function in L^q (q is defined by $1/p + 1/q = 1$), and so (1) shows that S_α is not complete.

Proof of III, $\Rightarrow$. That $\alpha \neq 0$ is obvious, since all the functions $\sin nx$ vanish at the origin. Suppose that $0 < \Re\alpha \leq 1$. Then $\left(\tan\frac{x}{2}\right)^{1-\alpha} \in L^1$, and so (1) shows that S_α is not complete.

Since the proof of III, $\Leftarrow$, is somewhat complicated and of a different spirit, we delay it until the end. We prove I, $\Leftarrow$, and II, $\Leftarrow$, now on the added assumption that $0 < \Re\alpha$ (the case $0 = \Re\alpha$ is a correspondence of III, $\Leftarrow$). Actually, I, $\Leftarrow$, is implied by II, $\Leftarrow$, but we give the proofs separately to illustrate the method.

Proof of I, $\Leftarrow$, for $0 < \Re\alpha$. We know by Theorem 2 that S_α has codimension at most 1 in the space C. By (1), this codimension is actually 1. Suppose we have an $f(x)$ such that

$$(11) \qquad f(x) \in L^\infty \quad \text{and} \quad \int_0^\pi f(x)\sin\left(nx + \frac{\pi}{2}\alpha\right) dx = 0 \qquad (n = 1, 2, \cdots).$$

Comparing (11) with (1) and keeping in mind that the codimension is 1, we must conclude that

$$(12) \qquad f(x) = C\left(\tan\frac{x}{2}\right)^{1-\alpha} \quad \text{a.e.}$$

We are assuming, however, that $\Re\alpha < 1$, and so $\left(\tan\frac{x}{2}\right)^{1-\alpha}$ is not in L^∞. Hence $C = 0$ in (12), and we deduce that $f(x) = 0$ a.e.

We have thus shown that (11) $\Rightarrow$ $f(x) = 0$ a.e., and this gives the required completeness.

Proof of II, $\Leftarrow$, for $0 < \Re\alpha$. By Theorem 2 and III, $\Rightarrow$, we know that S_α has codimension 1 in the space C. Suppose $\left(\text{with } \frac{1}{p} + \frac{1}{q} = 1\right)$ that

$$(13) \qquad f(x) \in L^q \quad \text{and} \quad \int_0^\pi f(x)\sin\left(nx + \frac{\pi}{2}\alpha\right) dx = 0 \qquad (n = 1, 2, \cdots).$$

Then $f(x)$ must be in L^1, and so (13) and (1), together with the fact that the codimension is 1, force us to conclude that

$$(14) \qquad f(x) = C\left(\tan\frac{x}{2}\right)^{1-\alpha} \quad \text{a.e.}$$

Now, in $\left[\frac{\pi}{2}, \pi\right]$,

$$\left|\left(\tan\frac{x}{2}\right)^{1-\alpha}\right|^q = \left(\tan\frac{x}{2}\right)^{(1-\Re\alpha)q} \geq \left(\tan\frac{x}{2}\right)^{\left(1-\frac{1}{p}\right)q} = \tan\frac{x}{2},$$

since $\Re\alpha \leq 1/p$. Because $\tan\frac{x}{2}$ is not integrable on $\left[\frac{\pi}{2}, \pi\right]$, we conclude that the constant C appearing in (14) must be 0, and so we further conclude that

(15) $$f(x) = 0 \text{ a.e.}$$

We have shown, then, that (13) ⇒ (15), and this gives the required completeness.

Proof of III, ⇐. Here we use (1) in a different way. Writing $1 - \alpha$ for α and $-x$ for x, we have the identities

(16) $$\int_{-\pi}^{0} \left|\tan\frac{x}{2}\right|^{\alpha} \cos\left(nx + \frac{\pi}{2}\alpha\right) dx = 0 \qquad (n = 1, 2, \cdots).$$

Consider the function $F(x)$ on $(-\pi, \pi)$ defined by

(17) $$F(x) = \begin{cases} 0 & \text{on } [0, \pi], \\ \left|\tan\frac{x}{2}\right|^{\alpha} & \text{on } (-\pi, 0). \end{cases}$$

By (16), $F(x)$ has a Fourier series of the form

$$F(x) \sim a + \sum_{n=1}^{\infty} c_n \sin\left(nx + \frac{\pi}{2}\alpha\right).$$

Later it will prove important to note that here

$$a = \frac{1}{2\pi}\int_{-\pi}^{0} \left|\tan\frac{x}{2}\right|^{\alpha} dx = \frac{1}{2\pi} B\left(\frac{1+\alpha}{2}, \frac{1-\alpha}{2}\right) = \frac{1}{2}\sec\frac{\pi}{2}\alpha,$$

so that

(18) $$a \neq 0.$$

Now introduce the function

(19) $$G(x) = \frac{\pi\varepsilon\cos\frac{x}{2}}{2\left|\sin\frac{x}{2}\right|^{1-\varepsilon}} \qquad (\varepsilon > 0).$$

We note that

$$\int_{-\pi}^{\pi} G(x)\,dx = \pi\varepsilon\int_{0}^{\pi} \frac{\cos\frac{x}{2}\,dx}{\left(\sin\frac{x}{2}\right)^{1-\varepsilon}} = 2\pi\int_{0}^{\pi} d\left(\sin\frac{x}{2}\right)^{\varepsilon} = 2\pi$$

and that $G(x)$ is even, so that the Fourier series for $G(x)$ can be written as

$$G(x) \sim 1 + 2\sum_{n=1}^{\infty} g_n \cos nx.$$

We form the convolution $F * G(x)$ by setting

(20) $$F * G(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) G(x - t)\, dt,$$

and we note that this has the Fourier series

(21) $$F * G(x) \sim a + \sum_{n=1}^{\infty} c_n g_n \sin\left(nx + \frac{\pi}{2}\alpha\right).$$

Next we observe that the L^1-modulus of continuity $\omega(\delta)$ of $G(x)$ satisfies an inequality $\omega(\delta) \leq C\,\delta^{\varepsilon}$, so that (since $F(x)$ is bounded) $F * G(x)$ belongs to Lip ε. By the Dini-Lipschitz test, then, this function has uniformly convergent Fourier series, and so we may improve (21) to the assertion that

(22) $a + \sum_{n=1}^{\infty} c_n g_n \sin\left(nx + \frac{\pi}{2}\alpha\right)$ converges uniformly to $F * G(x)$ on $[-\pi, \pi]$.

To estimate $F * G(x)$ on $[0, \pi]$, let $x \in (0, \pi)$. By (17), (19), (20), and integration by parts, we can write

$$F * G(x) = \frac{\varepsilon}{4} \int_{-\pi}^{0} \left|\tan\frac{t}{2}\right|^{\alpha} \frac{\cos\frac{x - t}{2}}{\left|\sin\frac{x - t}{2}\right|^{1-\varepsilon}}\, dt = \frac{\varepsilon}{2} \int_0^{\pi/2} (\tan u)^{\alpha} \frac{\cos\left(\frac{x}{2} + u\right)}{\left(\sin\left(\frac{x}{2} + u\right)\right)^{1-\varepsilon}}\, du$$

$$= \frac{\varepsilon}{2} \int_0^{\pi/2} \frac{(\tan u)^{\alpha}}{\sin u \cos u} \frac{\cos\left(\frac{x}{2} + u\right) \cos u \sin u}{\left(\sin\left(\frac{x}{2} + u\right)\right)^{1-\varepsilon}}\, du$$

$$= \frac{\varepsilon}{2\alpha} \int_0^{\pi/2} \frac{\cos\left(\frac{x}{2} + u\right) \cos u \sin u}{\left(\sin\left(\frac{x}{2} + u\right)\right)^{1-\varepsilon}}\, d(\tan u)^{\alpha}$$

$$= -\frac{\varepsilon}{2\alpha} \int_0^{\pi/2} (\tan u)^{\alpha}\, d\, \frac{\cos\left(\frac{x}{2} + u\right) \cos u \sin u}{\left(\sin\left(\frac{x}{2} + u\right)\right)^{1-\varepsilon}}.$$

Since α is imaginary, we can finally estimate this as follows. For $x \in (0, \pi)$,

(23) $$|F * G(x)| \leq \frac{\varepsilon}{2|\alpha|} V,$$

where V is the total variation of $\dfrac{\cos\left(\frac{x}{2} + u\right) \cos u \sin u}{\left(\sin\left(\frac{x}{2} + u\right)\right)^{1-\varepsilon}}$ on $\left[0, \frac{\pi}{2}\right]$.

By writing the logarithmic derivative of the fraction, we see that the equation for the singular points involves a fourth-degree trigonometric polynomial. Therefore there are at most eight singular points, and so

$$V \leq 16\,M, \tag{24}$$

where M is the maximum of $\left|\dfrac{\cos\left(\frac{x}{2}+u\right)\cos u \sin u}{\left(\sin\left(\frac{x}{2}+u\right)\right)^{1-\varepsilon}}\right|$ on $\left[0, \frac{\pi}{2}\right]$. Since $\left|\cos\left(\frac{x}{2}+u\right)\right| \leq 1$ and

$$\sin\left(\frac{x}{2}+u\right) = \sin\frac{x}{2}\cos u + \cos\frac{x}{2}\sin u \geq \left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)\min(\cos u, \sin u)$$

$$\geq \min(\cos u, \sin u) \geq \cos u \cdot \sin u,$$

we have the estimate

$$M \leq 1. \tag{25}$$

Combining (18), (23), (24), and (25), we obtain the estimate

$$\left|1 + \frac{1}{a}\sum_{n=1}^{\infty} c_n g_n \sin\left(nx + \frac{\pi}{2}\alpha\right)\right| \leq \frac{8\varepsilon}{|\alpha a|} \quad \text{in } [0, \pi].$$

By choosing N large enough and using (22), we deduce from this that

$$\left|1 + \frac{1}{a}\sum_{n=1}^{N} c_n g_n \sin\left(nx + \frac{\pi}{2}\alpha\right)\right| \leq \frac{9\varepsilon}{|\alpha a|} \quad \text{in } [0, \pi].$$

Since ε is arbitrary, this tells us that the function 1 is in the uniform span of the set S_α. But Theorem 2 states that the uniform span of $1 \cup S_\alpha$ is all of C, and the proof is complete.

Harvard University
Cambridge, Massachusetts 02138
and
Yeshiva University
New York, New York 10033