NONLINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES, AND APPLICATIONS

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1. INTRODUCTION

In two recent papers [3], [5], the author and J. N. Welch investigated the differential equation

$$(*) \qquad \frac{dx}{dt} = Ax + f(t, x)$$

in a Banach space X with regard to the existence, uniqueness, and stability of periodic solutions, almost-periodic solutions, compact solutions, and bounded solutions, together with the approximation of a compact solution in finite-dimensional subspaces and continuity of a solution relative to a parameter. For the case of an unbounded operator A, it was assumed, among other conditions, that f(t, x) lies in the domain of A.

The purpose of this note is to establish the results obtained in [3] and [5] without the assumption that f(t, x) lies in the domain of A, provided that A generates a holomorphic semigroup. To illustrate the application of these results, we use them and some recent results of K. Yosida [8], [9] on holomorphic semigroups to show that, under suitable conditions, the nonlinear diffusion equation

$$(**) \qquad \frac{\partial u}{\partial t} = a(x)\frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} + c(x)u + \sum_{i=0}^{n} d_i(t, x)u^i$$

admits a unique stable solution u(t, x) that is (a) periodic in t, (b) almost-periodic in t, (c) compact in t, or (d) bounded in t and is (i) periodic in x, (ii) almost-periodic in x, (iii) bounded and uniformly continuous in x, (iv) vanishing at infinity in x, (v) asymptotically periodic in x, or (vi) asymptotically almost-periodic in x. Thus there are solutions of 24 different types.

In [4], an equation similar to (**) was treated with a more general nonlinear term; but only mild solutions were established, and these are the limits of true solutions of certain approximate equations. The solutions of the equation (**) obtained in this note are true solutions. Recently, T. Kato reviewed the work that has been done on equations of the form (*) by means of semigroups; the readers are referred to his article [1] for further references.

2. THE MAIN RESULTS

Let R be the set of real numbers, R^+ the nonnegative real numbers. We make the following three assumptions.

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(1) The unbounded closed linear operator A maps the Banach space X into itself, and it is the infinitesimal generator of a semigroup $\{\exp(tA),\ t \geq 0\}$ of class $\{C_0\}$ such that

$$\|\exp(tA)\| \leq \beta e^{-\alpha t} \qquad \text{for all } t \geq 0,$$

where α and β are positive constants; moreover, the semigroup has a holomorphic extension in a sector of the complex plane containing the positive real axis [7, p. 254], and $\exp(-t)\exp(tA)$ is bounded there.

(2) f(t, x) is a mapping from $R \times X$ to X such that (a) f(t, x) is a B. U. L. function [3, p. 851] in t; (b) the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq \theta(t, \rho)\|x - y\| \qquad (\|x\| \leq \rho \text{ and } \|y\| \leq \rho)$$

holds for almost all t in R; here, for each fixed ρ, $\theta(t, \rho)$ is a real-valued B. U. L. function in t, and for each a in R it is bounded on some finite interval $a^* \leq t \leq a$; (c) for each fixed negative h in some neighborhood of 0, the inequality

$$\sup_{\|x\| \leq \rho} \|f(t + h, x) - f(t, x)\| \leq \phi(t, \rho)|h|^q$$

holds for almost all t in R, and for every fixed t in R it holds for almost all negative h in some neighborhood of 0; here q is a constant $(0 < q \leq 1)$; for each ρ, $\phi(t, \rho)$ is a real-valued B. U. L. function in t, and for each a in R it is bounded on some finite interval $a^* \leq t \leq a$; (d) for each a in R and each compact set C in X, there is some interval $a^* \leq t \leq a$ of t such that f(t, x) maps $[a^*, a] \times C$ into some compact set.

(3) for some positive numbers ρ and r $(r < 1)$,

$$\sup_{t \in R} \beta \int_{-\infty}^{0} \exp(\alpha s)\, \theta(t + s, \rho)\, ds < r,$$

$$\sup_{t \in R} \left\| \int_{-\infty}^{0} \exp(-sA)\, f(t + s, 0)\, ds \right\| < \rho(1 - r)/2.$$

In contrast to the situation in [3] and [5], it is not assumed here that f(t, x) lies in the domain of A. Instead, we have added 2(c), which requires that f(t, x) have a certain weak Hölder continuity from the left in t. These conditions allow f(t, x) to be discontinuous in t. Also, A now generates a holomorphic semigroup. In a recent note [2], C. I. Langenhop showed that the first condition in (3) implies that

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t \theta(s, \rho)\, ds < \frac{\alpha}{\beta},$$

and this was used in [3] to establish the stability of solutions. For the definition of a solution, see [3, p. 872].

THEOREM 1. *Let conditions* (1) *to* (3) *be satisfied for some fixed* ρ *and* r. *Then*

(i) *there exists exactly one bounded solution* x(t) *of the differential equation* (*) *satisfying the condition* $\|x(\cdot)\|_\infty \leq \rho$ *(and, in fact,* $\|x(\cdot)\|_\infty < \rho$);

(ii) x(t) *is negatively unstable; in fact, every other solution* y(t) *of* (*) *that exists for* $t \leq a$ *for some* a *and* $\|y(a)\| \leq \rho$ *must also satisfy* $\|y(t)\| > \rho$ *for infinitely many* t *without lower bounds;*

(iii) x(t) *is positively asymptotically stable; in fact, for each real number* a, *there exist two positive numbers* δ *and* ω *such that, for any other solution* y(t) *of* (*) *in* $[a, \infty)$, *each of the two conditions*

$$\text{(a)}\quad \|x(a) - y(a)\| < \delta, \qquad \text{(b)}\quad \|y(t)\| \leq \rho \text{ in } [a, a+\omega]$$

implies that $\|y(t)\| \leq \rho$ *in* $[a, \infty)$ *and* $\|x(t) - y(t)\| \to 0$ *exponentially as* $t \to \infty$;

(iv) x(t) *is periodic of period* 1 *if* f(t, x) *and* $\theta(t, \rho)$ *are periodic of period* 1 *in* t;

(v) x(t) *is almost-periodic if* f(t, x) *and* $\theta(t, \rho)$ *are generalized almost-periodic functions* [3, p. 851] *of* t;

(vi) x(t) *is compact if* f(t, x) *is a* $(\text{BULC})^-$ *function* [5, p. 272] *in* t.

Remark. For the cases (iv) and (v), the numbers δ and ω in (iii) can be chosen independent of the number a, as in [3, p. 867]. If, in addition, f is also jointly continuous in t and x, and θ and ϕ are bounded in finite intervals of t, then Ax(t) and dx(t)/dt are continuous in t. We should note that the theorem applies in the important special case where the term f(t, x) in (*) does not depend on x.

Proof. In the same way as in the proof of Theorem 1 in [3] (see also a remark there on p. 872), there exists a unique bounded continuous function x(t) on R to X satisfying the conditions

$$x(t) = \int_{-\infty}^{0} \exp(-sA)\, f(t+s, x(t+s))\, ds \qquad (-\infty < t < \infty) \tag{4}$$

and

$$\|x(\cdot)\|_\infty < \rho. \tag{5}$$

We now show that there exists a constant C such that

$$\|x(h+\cdot) - x(\cdot)\|_\infty \leq C\,|h|^q \quad \text{for all negative } h \text{ sufficiently near } 0. \tag{6}$$

In fact, from (4), (2b), (2c), and (5), we have for each such negative h the inequality

$$\begin{aligned} &\|x(h+t) - x(t)\| \\ &\leq \beta \int_{-\infty}^{0} \exp(\alpha s)\, \theta(t+s, \rho)\, ds\, \|x(h+\cdot) - x(\cdot)\|_\infty + \beta \int_{-\infty}^{0} \exp(\alpha s) \phi(t+s, \rho)\, ds\, |h|^q. \end{aligned} \tag{7}$$

This inequality together with (3) implies (6).

Let $0 < d^* < d$. Since A is closed and exp(dA) maps X into the domain of A [7, p. 254], the closed-graph theorem assures us that A exp(dA) is a bounded linear operator with domain X. It follows that

(8)
$$A \int_{-\infty}^{-d} \exp(-sA)\, f(t+s,\, x(t+s))\, ds = \int_{-\infty}^{-d} A \exp(-sA)\, f(t+s,\, x(t+s))\, ds \qquad (-\infty < t < \infty)$$

is a meaningful and true statement. Since A generates a holomorphic semigroup, $\|sA \exp(sA)\|$ is uniformly bounded in $0 < s \leq 1$ (see [7, p. 254]). Using this, (2b), (2c), and (6), and writing $f(t+s, x(t+s))$ in the form

$$f(t+s,\, x(t+s)) - f(t,\, x(t+s)) + f(t,\, x(t+s)) - f(t,\, x(t)) + f(t,\, x(t)),$$

we find that

(9)
$$\left\| \int_{-d}^{-d^*} A \exp(-sA) f(t+s,\, x(t+s))\, ds \right\| \leq C^* \int_{-d}^{-d^*} [\phi(t, \rho) + \theta(t, \rho)](-s)^{q-1}\, ds + \|[\exp(d^*A) - \exp(dA)] f(t,\, x(t))\|$$

for small d, where C^* is some constant independent of t, d, and d^*. Thus for each t the first integral in (9) converges to 0 as d and d^* tend to 0. Because A is a closed linear operator, we have proved that

x(t) lies in the domain of A and

(10)
$$A\, x(t) = \lim_{d \downarrow 0} \int_{-\infty}^{-d} A \exp(-sA)\, f(t+s,\, x(t+s))\, ds.$$

For each real number a, the functions $\theta(t, \rho)$ and $\psi(t, \rho)$ are by hypothesis bounded in some interval $[a^*, a]$ of t, and $f(t, x(t))$ maps $[a^*, a]$ into a compact set. It follows also that as d and d^* tend to 0, the first integral in (9) converges to 0 uniformly for all t in $[a^*, a]$. The continuity of each integral in (8) (see [3, p. 872]) them implies that

(11) $A\,x(t)$ is strongly measurable and continuous from the left.

From (9) and (10), we see that $\|A\,x(t)\|$ is dominated by a B. U. L. function, and hence

(12) $A\,x(t)$ is a B. U. L. function.

Using (10) and (11), we can verify in the same way as in [3, p. 873] that for almost all t the function x(t) has a derivative and satisfies the differential equation (*), and that $\exp\{(b-t)A\}\, x(t)$ is absolutely continuous on every finite interval [a, b]. Moreover,

(13) $\dfrac{dx(t)}{dt}$ is a B. U. L. function.

To prove that $x(t)$ is absolutely continuous on finite intervals, take a positive number a. Some calculations show that, for a positive number h,

$$\exp(aA)\{x(t+h) - x(t)\}$$

can be written as

$$(14) \quad -\int_{-h}^{0} \exp(-sA)\, A \exp(aA) x(t)\, ds + \int_{t}^{t+h} \exp\{(a+t+h-s)A\} f(s, x(s))\, ds,$$

and thus $\exp(aA)x(t)$ is absolutely continuous on finite intervals. Together with (13), this implies that

$$(15) \quad \exp(aA)\{x(t+h) - x(t)\} = \int_{t}^{t+h} \exp(aA) x'(s)\, ds = \exp(aA) \int_{t}^{t+h} x'(s)\, ds.$$

Letting $a \to 0$, we see that $x(t)$ is the indefinite integral (Bochner) of its derivative and hence is absolutely continuous on finite intervals. This completes the proof of part (i). In exactly the same way as in [3] and [5], we have parts (ii) to (vi).

It is easy to verify that $Ax(t)$ and $dx(t)/dt$ are continuous in t if, in addition, $f(t, x)$ is also jointly continuous in t and x, and $\theta(t, \rho)$ and $\phi(t, \rho)$ are bounded in finite intervals of t. (See the proof of (11), equation (*), and (15).)

By modifying the proof of part (i), we can establish the following local existence and uniqueness theorem.

THEOREM 2. *Let conditions* (1) *and* (2) *be satisfied, where* α *in* (1) *may be any real number. Then, for each number* a *and each vector* x_a *in the domain of* A, *there exists a unique solution* $x(t)$ *of the differential equation* (*), *on some interval* $[a, b]$, *that satisfies the condition* $x(a) = x_a$ *and varies continuously with* x_a. *The solution can be continued to the right as far as it remains bounded. If, in addition,* $f(t, x)$ *is continuous in* (t, x) *and* $\theta(t, \rho)$ *and* $\phi(t, \rho)$ *are bounded in* $a \leq t \leq b$, *then* $Ax(t)$ *and* $dx(t)/dt$ *are continuous in* $[a, b]$.

Proof. When such a solution $x(t)$ exists, it satisfies

$$(16) \quad x(t) = \exp\{(t-a)A\} x_a + \int_{a}^{t} \exp\{(t-s)A\} f(s, x(s))\, ds \qquad (a \leq t \leq b)$$

(see [3, p. 874]). On the other hand, by the usual method of successive approximation or by a contraction mapping, one can show that the integral equation (16) has a unique continuous solution $x(t)$ with $x(a) = x_a$, on some interval $[a, b]$, where b is any number greater than a such that

$$(17) \quad \|\exp\{(t-a)A\} x_a\| + \beta \int_{a}^{t} \exp\{(s-t)\alpha\} (\|f(s, 0)\| + \theta(s, \rho)\rho)\, ds < \rho$$

for all t in $[a, b]$, and where ρ is any fixed number larger than $\|x_a\|$. Clearly, by (17), $\|x(t)\| < \rho$ in $[a, b]$. Since the first term on the right of (16) has a continuous derivative in $[a, b]$ and $f(s, x(s))$ is bounded almost everywhere in $[a, b]$ by virtue of (2b) and (2c), it is easy to verify that there exists a constant C such that

$$\|x(t+h) - x(t)\|$$

(18) $$\leq C\,|h| + \left\| \int_{a-t-h}^{0} \exp(-sA)\,\{f(t+h+s,\, x(t+h+s)) - f(t+s,\, x(t+s))\}\, ds \right\|$$

for all sufficiently small negative h and all t such that $a \leq t+h < t \leq b$. Using the part of (17) that involves θ, we find (much as in (7)) that there exists a constant C such that

(19) $$\|x(t+h) - x(t)\| \leq C\,|h|^q \quad \text{for all sufficiently small negative } h$$

and for t such that $a \leq t+h < t \leq b$. Similarily, we prove that the integral in (16) (and thus $x(t)$) lies in the domain of A, that $Ax(t)$ is continuous from the left and Bochner-integrable in $[a, b]$, that for almost all t in $[a, b]$ the function $x(t)$ has a derivative and satisfies the equation (*), that $dx(t)/dt$ is Bochner-integrable in $[a, b]$, and that $\exp\{(b-t)A\}\,x(t)$ and $x(t)$ are absolutely continuous in $[a, b]$. The continuity of $x(t)$ relative to x_a and the continuation of $x(t)$ beyond b can be treated as in the classical case.

3. APPLICATIONS

We now apply the main results to the nonlinear diffusion equation (**). For convenience, we write

C_1 for the class of all real-valued continuous functions defined on R that are periodic with period 1;

C_2 for the class of all real-valued almost periodic continuous functions on R;

C_3 for the class of all real-valued, bounded, uniformly continuous functions defined on R;

C_4 for the class of all real-valued continuous functions on R that vanish at infinity;

C_5 for the class of all real-valued continuous functions on R that have the form $f+h$ ($f \in C_1$, $h \in C_4$);

C_6 for the class of all real-valued continuous functions on R that have the form $f+h$ ($f \in C_2$, $h \in C_4$).

The functions in C_6 are called asymptotically almost periodic, while those in C_5 are asymptotically periodic. It should be noted that the representation of a function in C_6 as a sum $f+h$ ($f \in C_2$, $h \in C_4$) is unique; the same holds in C_5. By the basic properties of almost periodic functions, it is easy to verify the following proposition:

(20) *The function spaces* $C_1, C_2, \cdots, C_6$ *are real commutative Banach algebras under the usual definition of addition, multiplication, multiplication by scalars, and the uniform norm. They are also closed under the operations of taking the maximum or the minimum of two functions.*

For each i ($i = 1, 2, \cdots, 6$), consider the linear operator A on a subset of C_i to C_i defined by

(21) $$(Au)(x) = a(x)\frac{d^2u(x)}{dx^2} + b(x)\frac{du(x)}{dx} + c(x)u(x),$$

where u, u', and u" belong to C_i, under the following condition:

(22) a(x), a'(x), b(x), and c(x) belong to C_i for each $i \neq 4$, and for $i = 4$ they are bounded, continuous, real-valued functions on R; moreover, a(x) has a positive lower bound a_0, and c(x) has a negative upper bound $-\alpha$; the domain D(A) of A consists of all functions u in C_i such that u' and u" are also in C_i.

Let $\hat{C}_i$ be the family of all complex-valued functions on R whose real and imaginary parts are functions in C_i. Just like the C_i, the $\hat{C}_i$ are complex commutative Banach algebras. Clearly, A is defined in each $\hat{C}_i$, and its domain D(A) consists of functions whose first and second derivatives are also in $\hat{C}_i$. We now show that

(23) A generates a positive holomorphic semigroup $\{\exp(tA), t \geq 0\}$ of class $\{C_0\}$ in $\hat{C}_i$, and

$$\|\exp(tA)\| \leq e^{-\alpha t} \qquad (t \geq 0).$$

For the space $\hat{C}_3$, this was verified by Yosida [9].

For the space $\hat{C}_2$, let us observe that the equation

(24) $$\lambda u(x) \pm a(x)\frac{du(x)}{dx} = f(x) \qquad (f \in \hat{C}_2,\ \lambda \text{ positive and constant})$$

has a unique solution u(x) in $\hat{C}_2$.

To see this, take the + sign in the equation. If there is a bounded solution u(x), then

(25) $$u(x) = \int_{-\infty}^{0} \exp\left(-\lambda \int_s^0 \frac{dr}{a(x+r)}\right) \frac{f(s+x)}{a(s+x)}\, ds \qquad (-\infty < x < \infty).$$

This can be verified by integration of the equation (24). On the other hand, we can verify directly that the expression on the right of (25) defines a function u(x) on R that satisfies (24). Moreover, u(x) is almost-periodic as we can see by using Bochner's criterion for almost-periodic functions. A similar treatment applies to the other case. Clearly,

(26) $$\|(\lambda I \pm a\, d/dx)^{-1} f\| = \|u\| \leq \|f\|/\lambda,$$

where I is the identity operator and $\|\cdot\|$ is the uniform norm. In the above argument, a can be replaced by $\sqrt{a}$. It follows from the theorem of Hille, Phillips, and Yosida [7] that the operator $\sqrt{a}\, d/dx$ generates a contraction group of operators of class $\{C_0\}$ on the space $\hat{C}_2$. The domain of this operator is of course the family of all functions in $\hat{C}_2$ whose first derivatives also lie in $\hat{C}_2$; it is dense in $\hat{C}_2$. Expressions similar to (25) show that

(27) $$(\lambda I \pm \sqrt{a}\, d/dx)^{-1} f \geq 0 \qquad \text{if } f \geq 0,$$

that is, these resolvents are positive operators. By a recent result of Yosida [9] (see also the proof there), $(\sqrt{a}\, d/dx)^2$ then generates a positive holomorphic contraction semigroup of class $\{C_0\}$ in $\hat{C}_2$. Express A in the form

$$A = \left(\sqrt{a(x)} \frac{d}{dx} \right)^2 + \left(b(x) - \frac{a'(x)}{2} + b^* \right) \frac{d}{dx} - b^* \frac{d}{dx} + c(x), \tag{28}$$

where the positive constant b^* is taken so large that the coefficient of the second term on the right of (28) is positive. Thus A can be considered as a sum of four operators. We can prove (exactly as in Yosida's paper [9]), using a recent perturbation theorem of Yosida [8] and a product formula of Trotter [6], that A generates a positive holomorphic contraction semigroup of class $\{C_0\}$. The above argument applies to the operator $(A + \alpha I)$ as well. It follows that the identity $\exp(tA) = \exp(-\alpha t) \exp\{t(A + \alpha I)\}$ implies the inequality in (23).

For other spaces, (23) can be verified either in exactly the same way as in $\hat{C}_2$ above, or as in $\hat{C}_3$ (see [9]). We omit the details. It should be noted that A, the resolvent $R(\lambda, A)$ at $\lambda > 0$, and thus $\exp(tA)$, being the strong limit of $(R(1, tA/n))^n$, all map C_i into C_i.

The result (23) implies that for each $f(x)$ in $\hat{C}_i$, $u(t, x) = (\exp(tA)f)(x)$ is the unique solution of the equation

$$\frac{\partial u}{\partial t} = Au \qquad \text{for } t > 0 \qquad (u(0, x) = f(x),\ f \in \hat{C}_i) \tag{29}$$

such that for each fixed $t \geq 0$, $u(t, x)$ is a function in $\hat{C}_i$. Moreover, u is infinitely differentiable with respect to t for positive t, and it has a Taylor expansion

$$u(t + h, x) = \sum_{n=0}^{\infty} \frac{1}{n!} h^n A^n u(t, x) \tag{30}$$

in $\hat{C}_i$, for each positive t and small h. Thus $u(t, x)$ is forwardly and backwardly unique, in the sense that the vanishing of $u(t, x)$ for all x at a fixed positive t implies the same at each $t \geq 0$, with the consequence that $f \equiv 0$. Since

$$\left(\frac{\partial^2}{\partial t^2} + A \right)^n u(t, x) = \left(\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} \right)^n u(t, x) \tag{31}$$

in $\hat{C}_i$ and is jointly continuous in (t, x) in $(0, \infty) \times R$, the theorem of Friedrichs' and Sobolev's lemma (see [7]) imply that $u(t, x)$ is a C^∞-function in $(0, \infty) \times R$ whenever condition (22) is satisfied and the coefficients a, b, and c in A are C^∞-functions in R.

Concerning the nonlinear terms in the equation (**), we now make the following four assumptions.

(32) (a) For each t in R, the coefficients $d_k(t, x)$ $(k = 0, 1, 2, \cdots, n)$ are functions in C_i for $i \neq 4$, and when $i = 4$, d_0 is in C_4 and $d_1, d_2, \cdots, d_n$ are real-valued bounded continuous functions in x. (b) For every h in some neighborhood of 0, the inequalities

$$|d_k(t + h, x) - d_k(t, x)| \leq \phi(t)\, |h|^q \qquad (k = 0, 1, 2, \cdots, n)$$

hold for all t in R, uniformly for x in R; here q is a constant $(0 < q \leq 1)$, and $\phi(t)$ is a B. U. L. function, bounded on finite intervals of t. (c) For each k, the uniform norm $\|d_k(t, \cdot)\|$ taken with respect to x in R is a B. U. L. function in t. (d) The function

$$f(t, u) = \sum_{k=0}^{n} d_k(t, \cdot)u^k(\cdot) \qquad (u(\cdot) \in C_i)$$

is clearly a continuous mapping from $R \times C_i$ to C_i; it satisfies a Lipschitz condition of the type described in (2), with a coefficient $\theta(t, \rho)$ which we can take to be a continuous B. U. L. function in t, by virtue of (b) and (c). Let condition (3) be satisfied, with $\beta = 1$ for some positive numbers ρ and r, where A is now defined by (21) and has the properties in (23).

Now it is easy to verify that the mapping $f(t, u)$ from $R \times C_i$ to C_i and the operator A in (21) satisfy the conditions (1), (2), and (3) when X is taken to be C_i and $\beta = 1$. By a solution of the equation (**) in an interval I we shall, for the sake of simplicity in summarizing the results, mean a mapping $u(t, \cdot) = u(t, x)$ from I into C_i such that it is a solution of (**) when the latter is considered as a differential equation in the Banach space C_i. Now Theorem 1 yields the following further result.

THEOREM 3. *For some* i $(i = 1, 2, \cdots, 6)$, *let condition* (32) *be satisfied by some* ρ. *Then, on* $-\infty < t < \infty$, *the differential equation* (**) *admits a unique uniformly bounded stable solution* $u(t, \cdot) = u(t, x)$ *in* C_i. *Moreover,*

(i) *if* f *and* θ *are periodic of period* 1 *in* t, *then* $u(t, \cdot)$ *is periodic of period* 1 *in* t,

(ii) *if* f *and* θ *are generalized almost-periodic functions in* t, *then* $u(t, \cdot)$ *is almost-periodic in* t,

(iii) *if* $f(t, u)$ *is compact in* t *for each* u *in* C_i, *then* $u(t, \cdot)$ *is compact in* t.

A local result of the type of Theorem 2 also holds for the equation (**). Clearly, condition (32) can be relaxed. In the important special case, where the equation (**) contains no nonlinear terms, we should note that condition (32) (under which Theorem 3 applies) becomes very simple.

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